# Producing Omori's law from stochastic stress transfer and release

Mark Bebbington, *Massey University*

(joint work with Kostya Borovkov,
*University of Melbourne*)

# Abstract

We present an alternative to the ETAS model. The continuous time two-node network stress release/transfer Markov model has one node (denoted by A) loaded by external tectonic forces at a constant rate, with `events' (mainshocks) occurring at random instances with risk given by a function of the `stress level' at the node. Each event adds (or removes) a proportional amount of stress to the second node (B), which experiences `events' in a similar way, but with another risk function (of the stress level at that node only). When that risk function satisfies certain simple conditions (it may, in particular, be exponential), the frequency of jumps (aftershocks) at node B, in the absence of any new events at node A, follows Omori's law for aftershock sequences. When node B is allowed tectonic input, which may be negative, i.e., aseismic slip, the frequency of events takes on a decay form that parallels the constitutive law derived by Dieterich (1994). This fits very well to the modified Omori law, with a wide possible variation in the p-value. We illustrate the model by fitting it to aftershock data from the Valparaiso earthquake of March 3 1985, and showing how it can be used in place of conventional `stacking' procedures to determine regional p-values.

# Variation in Aftershock Decay

The modified Omori aftershock formula is

$$n(t) = K(c+t)^{-p}$$

where $0.9 < p < 1.8$, differing from sequence to sequence.

Stochastic seismicity models have to cater for this variation in p, e.g., the ETAS model (Ogata, 1988). We construct an alternative model which also allows for more general background sequence behaviour.

# Linked Stress Release Model (LSRM)

(Liu *et al.*, 1999; Bebbington and Harte, 2003).

Based on a stochastic version of elastic rebound.

Space is divided into ‘regions’, with the stress (Benioff strain) in region i evolving as

$$X_i(t) = X_i(0) + \rho_i t - \Sigma_j \theta_{ij} S^{(j)}(t)$$

where $S^{(j)}$ is the cumulative stress release in region j, $\theta_{ij}$ can be positive (damping) or negative (excitatory), and $\rho_i$ is tectonic input.

# Point Process Intensity

Using a hazard function $\Psi(X) = \exp(\mu+\nu X)$, we get a point process conditional intensity

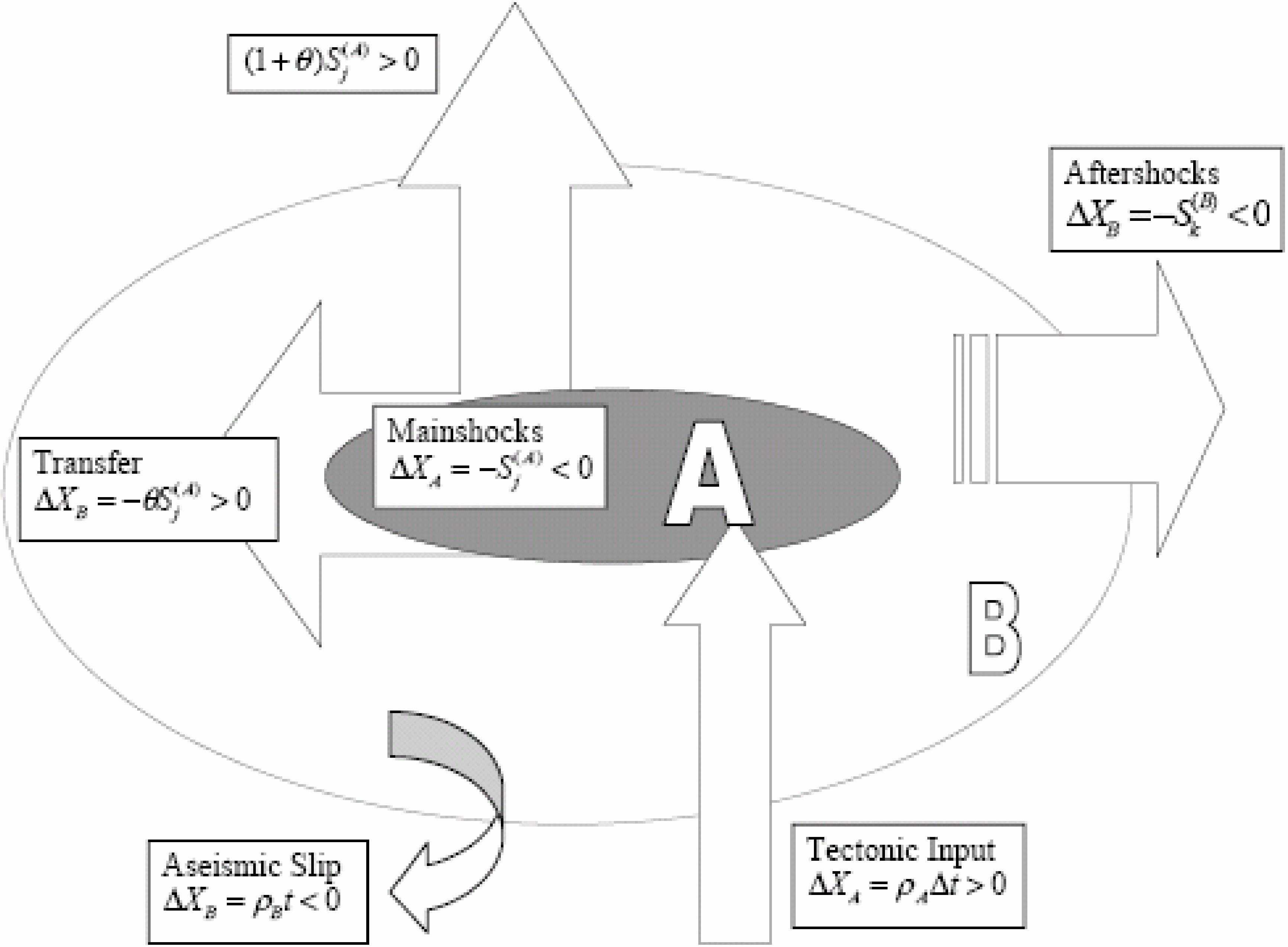
$$\lambda_i(t) = \exp[\alpha_i+\nu_i(\rho_i t - \Sigma_j\theta_{ij}S^{(j)}(t))]$$

which can be fitted by maximizing

$$\log L = \Sigma_i \left( \Sigma_k \log\lambda_i(t_k) - \int \lambda_i(t)dt \right)$$

where events occur at times $t_{k.}$

We will use a 2-region model, where region A represents mainshocks, and region B aftershocks.



$(1+\theta)S_j^{(A)} > 0$
Aftershocks
$\Delta X_B = -S_k^{(B)} < 0$
Transfer
$\Delta X_B = -\theta S_j^{(A)} > 0$
Mainshocks
$\Delta X_A = -S_j^{(A)} < 0$
A
B
Aseismic Slip
$\Delta X_B = \rho_B t < 0$
Tectonic Input
$\Delta X_A = \rho_A \Delta t > 0$

# Derivation of the Decay Formula

Let $X_A(t)$, $X_B(t)$ be the stress levels of the nodes.

As $\theta_{AB} = 0$ (no transfer from B to A), $X_A(t)$ is just a simple SRM, with known behaviour (Zheng, 1991; Borovkov and Vere-Jones, 2000)

$X_B(t)$ is also a simple SRM, except for occasional random increases due to transfer from A. Events following these can be interpreted as aftershocks.

# Analysis of Node B

Let Z(t) be the stress level at node B in the absence of further transfers from A. Then the 'typical behaviour' of the hazard $\Psi(Z(t))$ will be the frequency law for aftershocks.

If $\rho = \rho_B$, $\Psi = \Psi_B$, Z(t) has generator

$$\mathcal{A}h(z) = \lim_{\Delta \to 0} \Delta^{-1} ( E( h(Z(t+\Delta)) | Z(t)=z ) - h(z) )$$
$$= \rho h'(z) + \Psi(z) ( E\, h(z - \xi) - h(z) )$$

acting on the function h, where $\xi$ is a random variable for the jump down (a/s size) at node B

# Test Functions

For any 'test function' $\phi(z)$, $E\phi(X(t)) = E\phi(Y(t))$ iff $X(t)$ has the same distribution as $Y(t)$.

If the 'typical value' of $X(t)$ is a deterministic function $m(t)$, then $E\phi(X(t)) \approx \phi(m(t))$, defining the 'typical behaviour' of $X(t)$.

Choosing $\phi(z) = z^k$, $z > 0$, $k = \ldots -1,0,1,\ldots$, with $h(z) = \phi(\Psi(z))$, and defining $f_k(t) = E\Psi^k(Z(t))$, the generator yields

$$f_k'(t) = k\nu\rho\, f_k(t) + (q(k\nu) - 1)\, f_{k+1}(t), \quad k \neq 0$$

where $q(y) = Ee^{-y\xi}$

# Omori Law

Hence $f_{-1}'(t) = -\nu\rho\, f_{-1}(t) + (q(-\nu) - 1)$ for the mean reciprocal hazard $f_{-1}$, yielding the 'typical behaviour function'

$$f(t) = \frac{1}{f_{-1}(t)} = \frac{\psi}{e^{-st} + (1 - e^{-st})a\psi / s}$$

where $\psi = \Psi(Z(0))$, $a = q(-\nu)-1 = Ee^{\nu\xi} - 1 > 0$, $s = \nu\rho$.

When there is no tectonic input at node B, $\rho = s = 0$, and this reduces to

$$f(t) = \frac{\psi}{1 + a\psi t} = \frac{a^{-1}}{(a\psi)^{-1} + t}$$

which is the Omori law with $p = 1$.

# Variance of f(t)

$f_{-2}'(t) = (q(-2\nu) - 1)(\psi^{-1} + at/2)t + \psi^{-2}$, so

$$\mathrm{Var}\left(\frac{1}{\Psi(Z(t))}\right) = f_{-2}(t) - f_{-1}^2 = b\left(\frac{1}{\psi} + \frac{a}{2}t\right)t$$

where $b = q(-2\nu) – 1 - 2a = E(e^{-\nu\xi} - 1)^2$.

This is small when the initial risk $\psi$ is large and a is small (i.e., $\nu\xi$ is usually small), indicating that our \`typical behaviour' function is close to the observed rate of events on node B.

# Dieterich's Formula

The two node model produces a decay formula

$$f(t) = \frac{\psi}{e^{-st} + (1-e^{-st})a\psi/s} \qquad (1)$$

Dieterich (1994) derived the formula

$$R = \frac{r\dot{\tau}/\dot{\tau}_r}{\left[\frac{\dot{\tau}}{\dot{\tau}_r}\exp\left(\frac{-\Delta\tau}{A\sigma}\right)-1\right]\exp\left(\frac{-t}{t_a}\right)+1} \qquad (2)$$

By setting

$$s = \frac{1}{t_a}, \quad \psi = r\exp\left(\frac{\Delta\tau}{A\sigma}\right), \quad a = \frac{\dot{\tau}_r}{r\dot{\tau}t_a}$$

(1) and (2) are equivalent.

# Parameter equivalencies

$\nu\rho = s = 1/t_a$,

relating relaxation time and sensitivity/tectonic input rate

$\Psi(Z(0)) = \exp(\mu + \nu X(0)) = r \exp(\Delta\tau/A\sigma)$, or

$\mu = \ln r$ (reference seismicity rates), and

$\nu X(0) = \Delta\tau/A\sigma$,

where X(0) is the initial stress transfer from the mainshock

$E\xi \approx a/\nu = \rho e^{-\mu}\dot{\tau}_r/\dot{\tau} = \rho\dot{\tau}_r/r\dot{\tau}$

which gives the expected aftershock size

# Variation in p

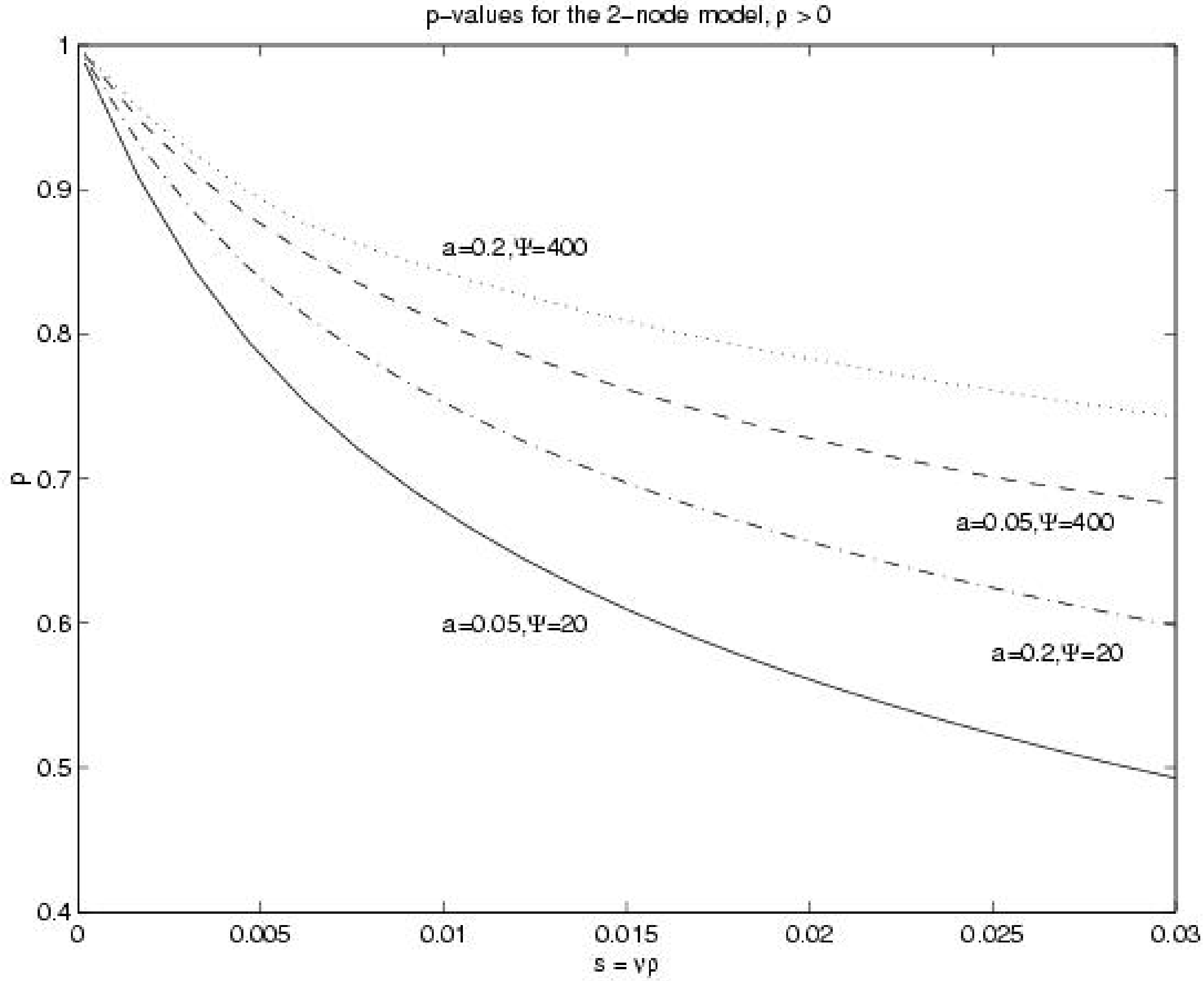
The 2-node formula

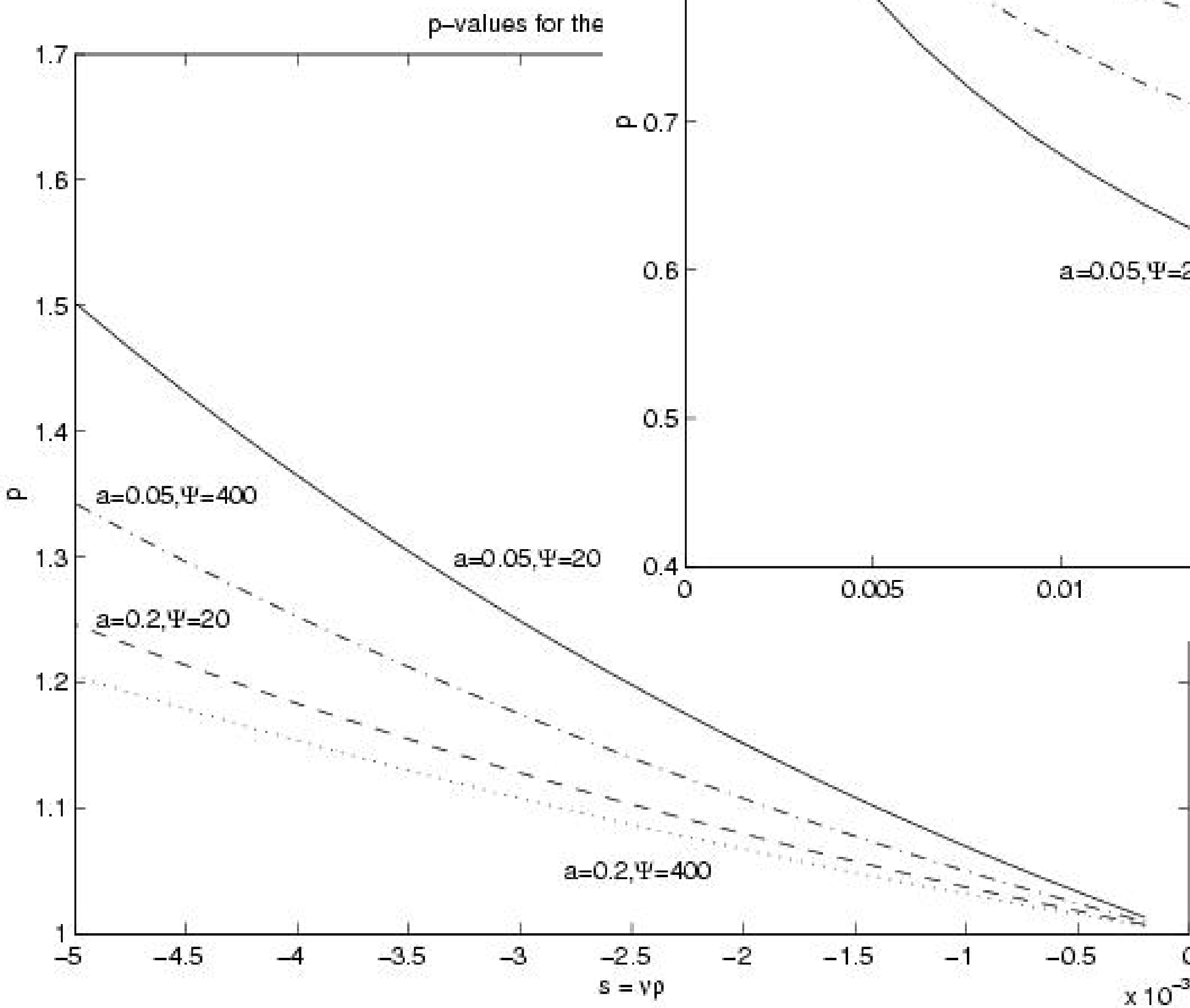
$$f(t) = \frac{\psi}{e^{-st} + (1-e^{-st})a\psi/s} \qquad (1)$$

gives the Omori law for $t < 1/s = t_a$, if $s > 0$ (D94).

However, ρ (and hence s ) can be negative, representing aseismic stress decrease. In this case, there is no background rate, as there is no steady state stress level, and $t_a < 0$ (actually undefined).

Numerically fitting (1) to the modified Omori law produces $0.5 < p < 1$ ($s > 0$) and $1 < p < 1.5$ ($s < 0$) for reasonable values of s, ψ and a.


p−values for the 2−node model, ρ > 0
a=0.2,Ψ=400
a=0.05,Ψ=400
a=0.05,Ψ=20
a=0.2,Ψ=20
p
s = νρ


p−values for the
a=0.05,Ψ=400
a=0.05,Ψ=20
a=0.2,Ψ=20
a=0.2,Ψ=400
p
s = νρ
x 10⁻³

# The Valparaiso Earthquake, 3.3.85

Aftershock sequence has 88 events of $M \geq 5$ in 802 days, with modified Omori formula $p = 1.038$.

Fitting the 2-node model, we find that

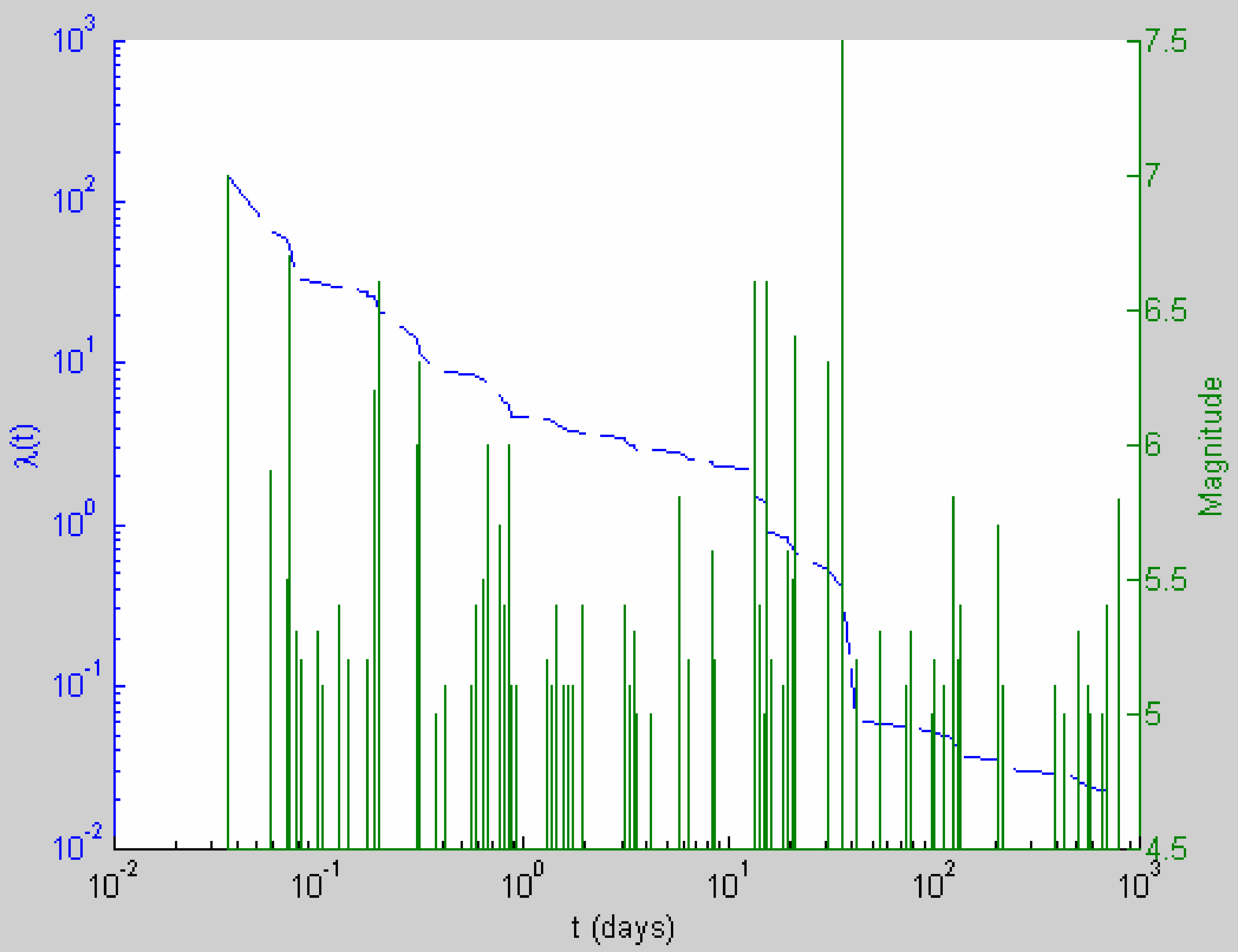
$$\psi = \psi(0) = e^{\alpha} = e^{4.9677} = 143.7$$

$$s = \nu\rho = 0.0248 \times (-0.0069) = -0.000172$$

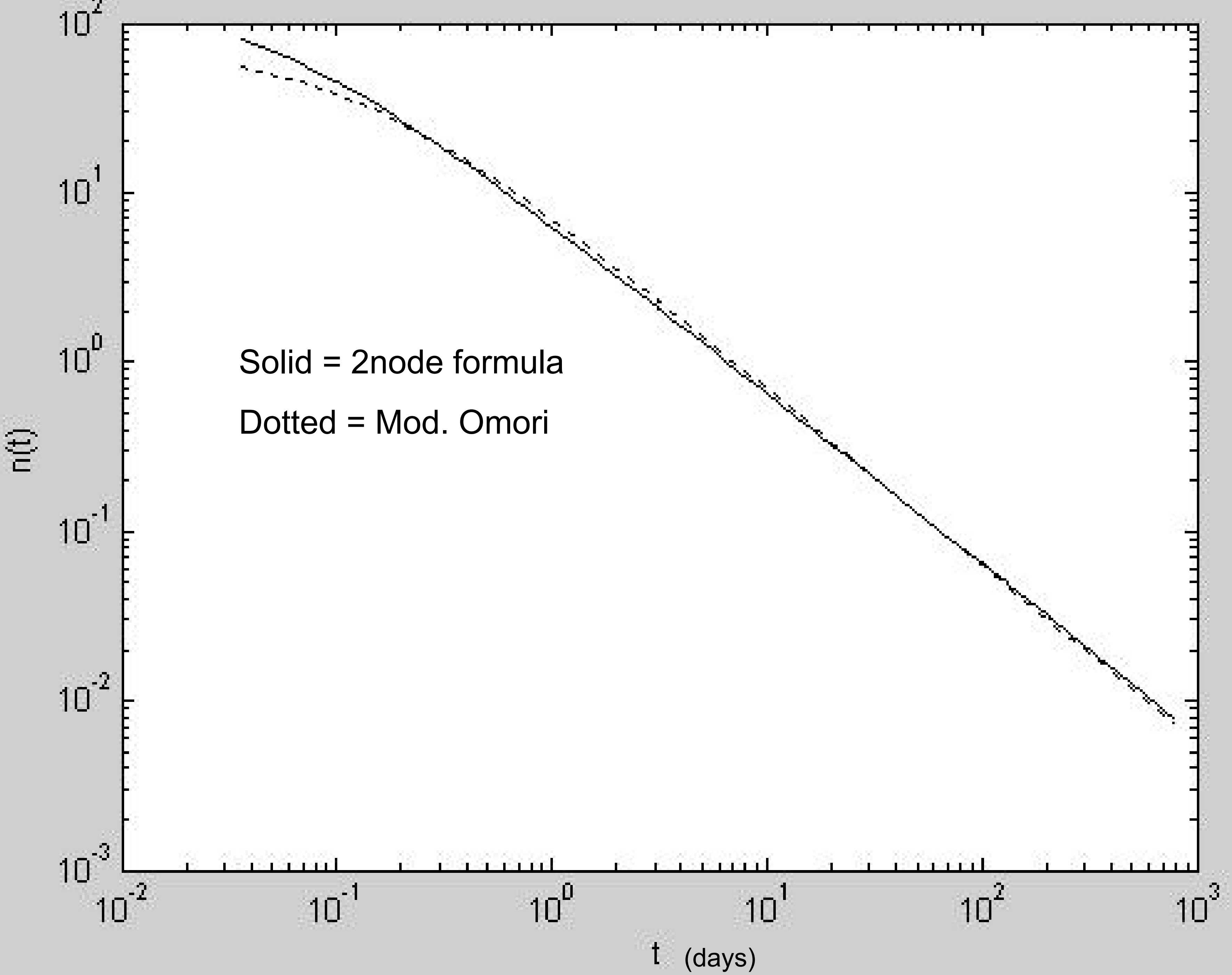
$$a = E(e^{\nu\xi} - 1)$$

$$= \Sigma_k \exp(0.0248 \times 10^{0.75(M_k - 5.0)})/88 - 1 = 0.153$$

Numerically matching the 2-node decay equation to the modified Omori formula then gives $p = 1.046$



$10^3$
$10^2$
$10^1$
$10^0$
$10^{-1}$
$10^{-2}$
λ(t)
7.5
7
6.5
6
5.5
5
4.5
Magnitude
$10^{-2}$
$10^{-1}$
$10^0$
$10^1$
$10^2$
$10^3$
t (days)



Solid = 2node formula
Dotted = Mod. Omori
n(t)
t (days)
10^2
10^1
10^0
10^-1
10^-2
10^-3
10^-2
10^-1
10^0
10^1
10^2
10^3

# Aftershock Stacking

Felzer et al. (2003) estimate the p-value for California by scaling and then stacking aftershock sequences. The main sequences considered were

| Earthquake | Date | Mag. (PDE) | # $M \geq 4.8$ aftershocks in (0.02,180) days |
|---|---|---|---|
| Coalinga | 2.5.83 | 6.7 | 11 |
| Morgan Hill | 24.2.84 | 6.2 | 0 |
| N. Palm Springs | 8.7.86 | 6.1 | 0 |
| Oceanside | 13.7.86 | 5.8 | 0 |
| Loma Prieta | 18.10.89 | 7.1 | 8 |
| Joshua Tree | 23.4.92 | 6.3 | 3 (66 days only) |
| Landers | 28.6.92 | 7.6 | 29 |
| Northridge | 17.1.94 | 6.8 | 11 |
| Hector Mine | 16.10.99 | 7.4 | 6 |

# p-value for California

Fitting the two node (7 param.) model, we get a ΔAIC of 1.85 over the full (8 param.) model, with estimates

$$\alpha = \begin{pmatrix} -6.25 \\ -4.27 \end{pmatrix}, \nu = \begin{pmatrix} 0.016 \\ 0.089 \end{pmatrix}, \rho = \begin{pmatrix} 0.041 \\ -0.027 \end{pmatrix}, \Theta = \begin{pmatrix} 1 & 0 \\ -0.983 & 1 \end{pmatrix}$$

Now

$$a = \text{ave.}_{\text{a/s}}\left(\exp\left(0.089 \times 10^{0.75(\text{Mag} - 4.8)}\right)\right) - 1 = 0.619$$

$$\Psi = \text{ave.}_{\text{m/s}}\left(\exp\left(0.089 \times 0.983 \times 10^{0.75(\text{Mag} - 4.8)}\right)\right) = 7216.4$$

and $s = -0.027 \times 0.089 = -0.0024$.

This gives p = 1.074, compared to p = 1.08 from Felzer et al. Kisslinger and Jones (1991) estimate p = 1.11 for Southern California.

# Summary

- If there is no aseismic stress decrease, the 2-node model can reproduce Dieterich's formula, and thus Omori's law.
  - This gives a range of p-values $\leq 1$.
- If there is aseismic stress decrease, the 2-node model fits well to the modified Omori law with $p > 1$.
  - Fitting the SRM to an aftershock sequence confirms that the model decays correctly.
- The 2-node model provides a regional-scale model for both main sequence and aftershock events.
- The 2-node model provides an alternative to existing methods of `stacking' aftershocks for regional analyses.

# References


Bebbington, M. and Harte, D. (2003), The linked stress release model for spatio-temporal seismicity: formulations, procedures and applications. *Geophys. J. Int.,* **154,** 925-946.

Borovkov, K. and Bebbington, M. (2003), A stochastic two-node stress transfer model reproducing Omori's law. *Pure Appl. Geophys.*, **160,** 1429-1455.

Borovkov, K., and Vere-Jones, D., (2000), Explicit formulae for stationary distributions of stress release processes, *J. Appl. Probab.*, **37,** 315-321.

Dieterich, J., (1994), A constitutive law for rate of earthquake production and its application to earthquake clustering, *J. Geophys. Res.*, **99,** 2601-2618.

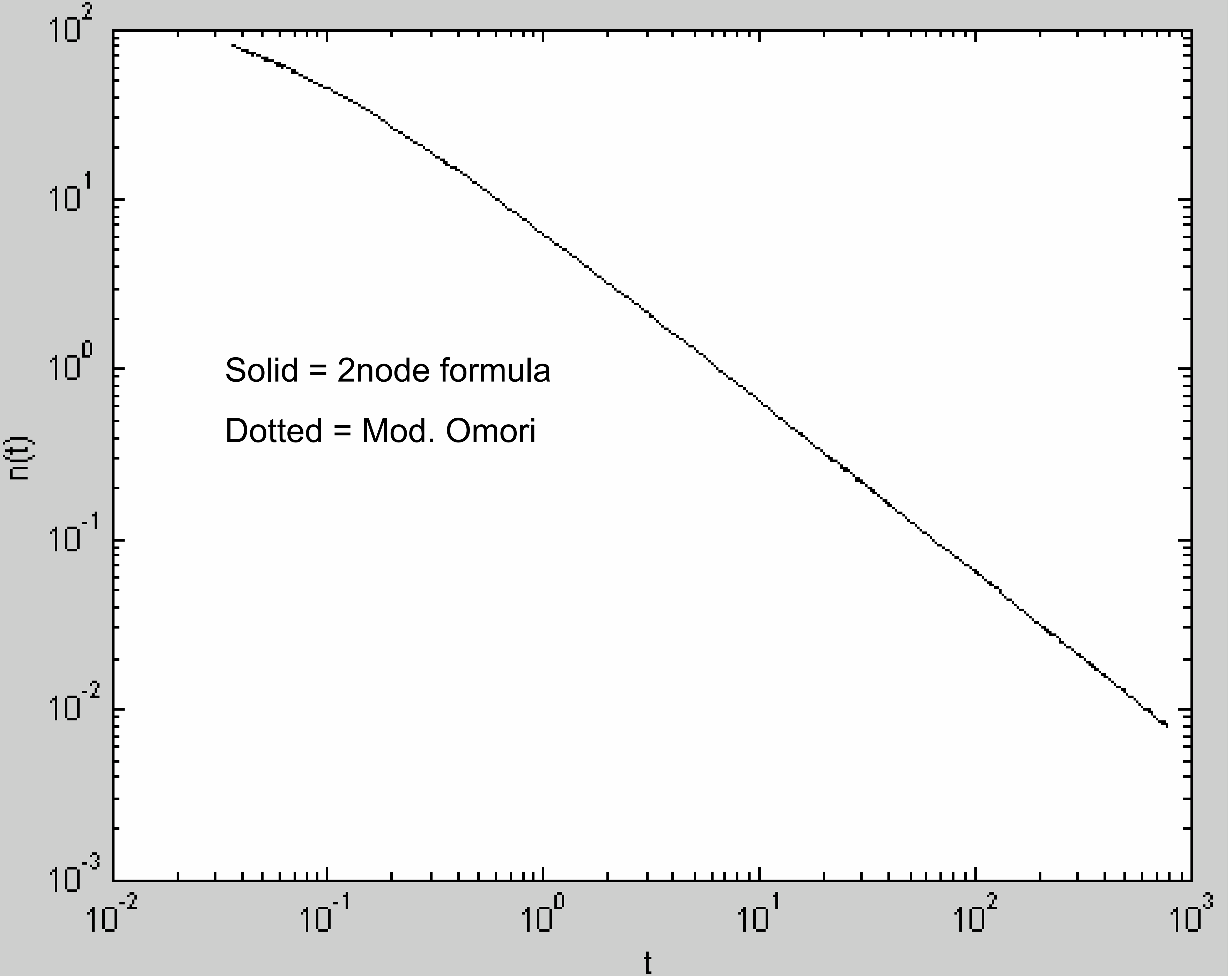
Felzer, K.R., Abercrombie, R.E. and Ekstrom, G., (2003), Secondary aftershocks and their importance for aftershock forecasting, *Bull. Seismol. Soc. Amer.,* **93,** 1433-1448.

Kisslinger, C. and Jones, L.M., (1991), Properties of aftershock sequences in southern California, *J. Geophys. Res.*, **96,** 11947-11958.

Liu, J., Chen, Y., Shi, Y. and Vere-Jones, D. (1999), Coupled stress release model for time-dependent seismicity, *Pure Appl. Geophys.,* **155,** 649-667

Ogata, Y., (1988), Statistical models for earthquake occurrences and residual analysis for point processes, *J. Amer. Statist. Assoc.,* **83,** 9-27.

Zheng, X., (1991), Ergodic theorems for stress release processes, *Stoch. Proc. Appl.,* **37,** 239-258.



Solid = 2node formula
Dotted = Mod. Omori
n(t)
t
10^2
10^1
10^0
10^-1
10^-2
10^-3
10^-2
10^-1
10^0
10^1
10^2
10^3