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Symmetry and chaos in the complex Ginzburg–Landau equation— I. Reflectional symmetries

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Abstract. *The complex Ginzburg–Landau (CGL) equation on a one-dimensional domain with periodic boundary conditions has a number of different symmetries. Solutions of the CGL equation may or may not be fixed by the action of these symmetries. We investigate the stability of chaotic solutions with some reflectional symmetry to perturbations which break that symmetry. This can be achieved by considering the isotypic decomposition of the space and finding the dominant Lyapunov exponent associated with each isotypic component. Our numerical results indicate that for most parameter values, chaotic solutions that have been restricted to lie in invariant subspaces are unstable to perturbations out of these subspaces, leading us to conclude that for these parameter values arbitrary initial conditions will generically evolve to a solution with the minimum amount of symmetry allowable. We have also found a small region of parameter space in which chaotic solutions that are even are stable with respect to odd perturbations.*

1 Introduction

Pattern formation in non-linear partial differential equations (PDEs) is a much studied topic. One common problem is determining the spatial patterns which can occur when a spatially uniform (steady) state loses stability. Yamada and Fujisaka (1983) were interested in the stability of spatially uniform 'chaotic' solutions of a non-linear PDE to perturbations which are not spatially uniform. In order to study this problem, they considered a finite-difference discretization of the PDE which gave a finite-dimensional system of coupled oscillators. The uniform state for the PDE corresponds to a synchronized state for the coupled oscillators. Stability of this uniform state was described in terms of what we now call normal Lyapunov exponents. However, the only numerical results presented in this work were for two coupled Lorenz systems.

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This work went largely unnoticed until Pecora and Carroll (1990) demonstrated that in some circumstances it is possible to synchronize two identical chaotic systems by linking them with a common signal. Since that time, there has been much interest in and study of synchronization in systems of coupled oscillators, one interesting application being secure communication (Ogorzalek, 1993).

Mathematically speaking, synchronization corresponds to motion in an invariant subspace which is stable with respect to perturbations normal to the subspace. If the largest normal Lyapunov exponent, associated with perturbations normal to the subspace, is negative, then the synchronized state has a positive measure basin of attraction, which may however be riddled so that there is a dense set of positive measure in any neighbourhood of the invariant subspace which is in the basin of another attractor (Alexander *et al.*, 1992; Ashwin *et al.*, 1994, 1996; Ott & Sommerer, 1994). If the largest normal Lyapunov exponent changes sign as a parameter is varied, then a blowout bifurcation occurs which may be either supercritical or subcritical (Ashwin *et al.*, 1994). In order to gain a deeper understanding of these phenomena, model equations of low dimension are often studied (Ashwin *et al.*, 1998).

The synchronized state of coupled identical oscillators is a natural setting for an invariant subspace. Another natural way of generating invariant subspaces is by the use of symmetry. Fixed point spaces are invariant under the dynamics of a system with symmetry and blowout bifurcations can be considered from these invariant subspaces also. Rings of coupled oscillators with symmetry were considered by Aston and Dellnitz (1995) and it was shown how the normal Lyapunov exponents can be classified according to the symmetry of the problem. This also leads to more efficient methods of computing Lyapunov exponents since the linearization of the system which is used to compute them can be decomposed on the isotypic components which are associated with the different irreducible representations of the group action.

We now extend the ideas of Aston and Dellnitz (1995) to PDEs with symmetry. Again, the flow of a PDE is invariant on various fixed point spaces and, if this motion is chaotic, then we can determine normal Lyapunov exponents associated with different normal directions, which can be determined using the symmetry. Thus, we are returning to the problem which originally motivated the work of Yamada and Fujisaka (1983) but in a more general framework. Blowout bifurcations in PDE mean field dynamo models associated with a simple reflectional symmetry have also been considered in Covas *et al.* (1997, 1998).

The equation that we will apply these ideas to is the complex Ginzburg–Landau (CGL) equation

$$A_t = RA + (1 + i\nu)\nabla^2 A - (1 + i\mu)A|A|^2 \tag{1}$$

with $A \in \mathbb{C}$ and $R, \nu, \mu \in \mathbb{R}$. This equation was originally derived as a generic amplitude equation which describes the motion near to points of instability in equations which model fluid dynamics (Newell & Whitehead, 1969; Stewartson & Stuart, 1971; Stuart & DiPrima, 1978) and chemical turbulence (Kuramoto & Koga, 1981). It has also been studied widely as an interesting model with a rich solution structure as it is known to have a finite-dimensional attractor and inertial manifolds (Constantin, 1989; Doering *et al.*, 1988; Duan *et al.*, 1993). For some parameter values, it is known to show complicated spatio-temporal behaviour, which we will refer to as 'chaos'. We consider only the cubic non-linearity and one space dimension so that the solutions of (1) exhibit soft turbulence (Bartuccelli *et al.*, 1990).

The CGL equation (1) is equivariant with respect to various symmetry groups and its solutions (chaotic or otherwise) may possess one or more of these symmetries. The aim of this work is to numerically investigate solutions of the CGL equation which have various symmetries and study their stability or instability with respect to perturbations which break the symmetry of the solutions.

The Lyapunov exponents of the CGL equation are also of interest for other reasons. In particular, rigorous bounds on the dimension of the attractor have been obtained by bounding the Lyapunov dimension which is derived from the Lyapunov exponents via the Kaplan–Yorke formula (Bartuccelli *et al.*, 1990; Doering *et al.*, 1987, 1988). Further analytical results for the CGL equation have been obtained in Bartuccelli *et al.* (1996) and Doering *et al.* (1994).

In Section 2, we describe the symmetries of the CGL equation while in Section 3, we briefly review previous results regarding Lyapunov exponents and symmetry. We also show that the CGL equation may have three zero Lyapunov exponents due to the continuous symmetries of the equation. In Section 4, we describe the reflectional symmetries of the solutions that we consider in this paper while Sections 5 and 6 describe the numerical methods used and the results obtained. Finally, some conclusions are drawn in Section 7.

2 Symmetries of the CGL equation

In this section we briefly outline some of the theory of dynamical systems with symmetry, concentrating on its applicability to the CGL equation. Group theory is the natural language with which to discuss symmetry; see Golubitsky *et al.* (1988) for many results concerning the application of symmetry to dynamical systems and their bifurcations.

We consider a general evolution equation of the form

$$A_t = g(A), \qquad g: X \to X \tag{2}$$

where g is assumed to be a non-linear operator involving spatial derivatives and X is an appropriate Hilbert space which incorporates the boundary conditions. We also assume that g satisfies the equivariance condition

$$\gamma g(A) = g(\gamma A) \qquad \text{for all } \gamma \in \Gamma \tag{3}$$

where Γ is a compact Lie group. For any subgroup Σ of Γ, we define the fixed point space

$$\mathrm{Fix}(\Sigma) = \{A \in X : \sigma A = A \text{ for all } \sigma \in \Sigma\}$$

It is easily verified that if g satisfies the equivariance condition (3), then

$$g: \mathrm{Fix}(\Sigma) \to \mathrm{Fix}(\Sigma)$$

for all subgroups Σ of Γ and this implies that the fixed point spaces are invariant under the flow of the non-linear equation (2).

Suppose that a linear operator L commutes with Γ so that

$$\gamma L = L\gamma \qquad \text{for all } \gamma \in \Gamma \tag{4}$$

In this case, there are many more invariant subspaces for the linear operator L which involve the isotypic components of the space X, which we now describe.

The space X can be decomposed as a direct sum of irreducible subspaces

$$X = \sum_i \oplus V_i$$

If we group together all the V_i on which Γ acts isomorphically, then we obtain the 'Γ-isotypic decomposition'

$$X = \sum_k \oplus W_k$$

where each isotypic component W_i is the sum of isomorphic irreducible subspaces which are associated with one of the irreducible representations of the group Γ. Moreover, this decomposition is unique (Aston, 1991; Healey, 1989; Werner, 1990).

The significance of this isotypic decomposition is that for a linear operator L satisfying (4), all the isotypic components are invariant under L, that is

$$L : W_k \to W_k$$

This results in a block diagonal structure to the linear operator L.

We assume that W_1 is associated with the trivial irreducible representation $\gamma = I$ for all $\gamma \in \Gamma$ and so $W_1 = \mathrm{Fix}(\Gamma)$.

This is relevant to the calculation of Lyapunov exponents since the variational equation involves the linear operator $g_A(A)$. It is well known and easily verified that if there is a trajectory $A(t)$ of (2) such that $A(t) \in \mathrm{Fix}(\Sigma)$ for some subgroup Σ of Γ, then

$$\sigma g_A(A(t)) = g_A(A(t))\sigma \qquad \text{for all } \sigma \in \Sigma$$

Thus, the linear operator g_A decomposes on the Σ-isotypic components of the space X. The important symmetry group in this case is not the symmetry group Γ of the equation but the subgroup Σ of symmetries of the particular solution being considered.

We now consider the CGL equation (1) on the one-dimensional domain $[0, 2\pi)$ with periodic boundary conditions. This equation has both continuous and discrete symmetries which are given by

$$\begin{aligned} \theta A(x,t) &= e^{i\theta} A(x,t), && \theta \in [0, 2\pi) \\ r_\alpha A(x,t) &= A(x+\alpha, t), && \alpha \in [0, 2\pi) \\ \tau_\beta A(x,t) &= A(x, t+\beta), && \beta \in \mathbb{R} \\ s_1 A(x,t) &= A(-x, t) \end{aligned}$$

These symmetries correspond to a rotation of the complex amplitude, space translation, time translation and a spatial reflection, respectively. We note that a special case of the rotation occurs when $\theta = \pi$ and this gives another symmetry of order two. Since this will be important in our later work, we define

$$\pi A(x,t) := s_2 A(x,t) = -A(x,t)$$

Relative equilibria are associated with continuous symmetries and in this case, the θ symmetry gives rise to such solutions which are often referred to as rotating waves. These were studied in some detail in Doering *et al.* (1988) which included a linear stability analysis.

3 Lyapunov exponents and symmetry

The way that symmetry affects the determination of Lyapunov exponents was considered in Aston and Dellnitz (1995) and applied to systems of coupled oscillators. We briefly review the main results of that work and will then apply the ideas to the CGL equation.

3.1 Classification of Lyapunov exponents

For a general n-dimensional ordinary differential equation (ODE)

$$\dot{x} = f(x), \qquad x(0) = x_0 \tag{5}$$

we find the Lyapunov exponents by integrating the variational equation

$$\dot{\Phi} = Df(x(t))\Phi, \qquad \Phi(0) = I \tag{6}$$

where $Df(x(t))$ is the Jacobian of f evaluated at $x(t)$, the solution of (5). We define

$$\Lambda = \lim_{t\to\infty} [\Phi(t)^{\mathrm{T}}\Phi(t)]^{1/2t} \tag{7}$$

provided this limit exists, where the superscript 'T' denotes the matrix transpose. The Multiplicative Ergodic Theorem of Oseledec (Oseledec, 1968; Eckmann & Ruelle, 1985) states that this limit exists for μ-almost all x_0, where μ is the invariant measure associated with the attractor of (5). If the eigenvalues of Λ are m_i, $i = 1, 2, \ldots, n$, then the Lyapunov exponents of the solution $x(t)$ are

$$\lambda_i = \log|m_i|, \qquad i = 1, 2, \ldots, n$$

If (5) is equivariant with respect to some compact Lie group Γ, then we have the following result (Aston & Dellnitz, 1995).

Lemma 1. Let S be an invariant set contained in $\mathrm{Fix}(\Sigma)$ for some subgroup Σ of Γ. For $x_0 \in S$, let $x(t)$ be the solution of (5) with $x(0) = x_0$. Then the solution $\Phi(t)$ of the variational equation (6) commutes with the action of Σ, i.e.

$$\sigma\Phi(t) = \Phi(t)\sigma$$

for all $\sigma \in \Sigma$ and $t \geqslant 0$.

A corollary of this is that the matrix Λ defined by (7) also commutes with the action of Σ. Thus, Λ can be put into block diagonal form and so its eigenvalues, and thus the Lyapunov exponents, can be associated with particular isotypic components.

The dominant (most positive) Lyapunov exponent associated with each isotypic component is the most important one for our purposes since it indicates whether the invariant set S is stable with respect to perturbations associated with the particular isotypic component (a positive dominant Lyapunov exponent implies that S is unstable to such perturbations). Using the block diagonal form of Λ these are easily computed.

For a particular Σ-isotypic component W_k we know that $Df(x(t))$ leaves W_k invariant and so we denote its restriction to W_k by $D_k f(x(t)) : W_k \to W_k$. The dominant Lyapunov exponent associated with the isotypic component W_k can then be found using the vector form of the variational equation restricted to W_k given by

$$\dot{\phi}_k = D_k f(x(t))\phi_k, \qquad \phi_k(0) = w_k \in W_k \tag{8}$$

and is given by

$$\lambda_{1,k} = \lim_{t\to\infty} \frac{1}{t} \log \| \phi_k(t) \|$$

Note that only the vector equation needs to be integrated to find the dominant Lyapunov exponent as a randomly chosen initial condition $\phi_k(0) = w_k$ will swiftly line up with the direction in phase space of greatest expansion or least contraction.

These ideas generalize naturally to the evolution of PDEs defined on appropriate Hilbert spaces. In particular, the isotypic decomposition for PDEs is well defined and any linear partial differential operator which commutes with all the elements of a group leaves the isotypic components associated with that group invariant (Aston, 1991). Thus, the linear systems defined by (8) are also well defined when f is a partial differential operator. Finally, the dominant Lyapunov exponent for a PDE can be defined in terms of the rate of asymptotic exponential growth of the length of a vector (Doering *et al.*, 1988) and is defined by

$$\lambda_{1,k} = \limsup_{t\to\infty} \frac{1}{t} \log\{\sup_{x_0} \sup_{\|\phi_k(0)\| \leqslant 1} \| \phi_k(t) \| \}$$

where $\phi_k(t)$ is the solution of the variational equation (8) and x_0 is the initial condition for the PDE. Thus, the whole framework described in this section applies to PDEs as well as to ODEs.

3.2 Continuous symmetries

As mentioned in Section 2, the CGL equation has three continuous symmetries, namely θ, r_α and τ_β. It was shown in Theorem 2.15 of Aston and Dellnitz (1995) that continuous symmetries give rise to zero Lyapunov exponents provided that the continuous symmetry acts on the solution non-trivially. This result is based on the fact that if $x(t)$ is a trajectory of (5), then $Lx(t)$ is a solution of the variational equation for all $L \in \mathscr{L}$ where $\mathscr{L}$ is the Lie algebra of Γ. If a one-parameter subgroup acts trivially on the trajectory $x(t)$ and if $L \in \mathscr{L}$ is the corresponding element of the Lie algebra, then $Lx(t) = 0$ and so in this case, there is no zero Lyapunov exponent associated with this symmetry. This is a generalization of the well-known result that autonomous differential equations have one zero Lyapunov exponent which is due to the continuous time translation symmetry.

We denote by L_θ, L_α and L_β the elements of the Lie algebra associated with the θ, r_α and τ_β symmetries, respectively. Since the Lie algebra is the tangent to the group orbit at the origin, we have that

$$L_\theta A(x,t) = \left.\frac{\mathrm{d}(\theta A(x,t))}{\mathrm{d}\theta}\right|_{\theta=0} = \left.\frac{\mathrm{d}(\mathrm{e}^{\mathrm{i}\theta} A(x,t))}{\mathrm{d}\theta}\right|_{\theta=0} = \mathrm{i}A(x,t)$$

Similarly

$$L_\alpha A(x,t) = A_x(x,t), \qquad L_\beta A(x,t) = A_t(x,t)$$

Thus, any chaotic attractor which is not spatially uniform has three zero Lyapunov exponents associated with it. If the attractor lies in $\mathrm{Fix}(\Sigma)$ for some subgroup Σ of Γ, then the zero Lyapunov exponents may occur in different Σ-isotypic components. These can be determined by finding which isotypic components contain the trajectories $LA(x, t)$ for each $L \in \mathscr{L}$. However, for any symmetry, $L_\theta A(x, t)$ and $L_\beta A(x, t)$ will always have the same symmetry as the solution trajectory and so occur in $W_1 = \mathrm{Fix}(\Sigma)$ whereas in some cases, $L_\alpha A(x, t)$ may occur in a different isotypic component.

As a simple example, consider the CGL equation with homogeneous Neumann boundary conditions which is equivalent to considering solutions which are invariant under the reflectional symmetry s_1. Thus, the symmetries of the solution are given by $\Sigma = \{I, s_1\} \simeq \mathbb{Z}_2$. The Σ-isotypic components are $W_1 = \mathrm{Fix}(\Sigma)$ which consists of all even periodic functions and W_2 which consists of all odd periodic functions. When only a reflectional symmetry is involved, the isotypic components W_1 and W_2 are often referred to as the symmetric and antisymmetric spaces, respectively. In this case we see that $L_\theta A(x, t)$ and $L_\beta A(x, t)$ are symmetric functions while $L_\alpha A(x, t)$ is antisymmetric. Thus, there will be two zero Lyapunov exponents associated with the motion in $\mathrm{Fix}(\Sigma)$ arising from the rotational and time translation symmetries and one in the symmetry-breaking antisymmetric direction arising from the space translation symmetry. Since we numerically calculate only the largest Lyapunov exponent, this means that neither of the dominant Lyapunov exponents associated with the symmetric and antisymmetric spaces can be negative.

These theoretical predictions are consistent with the numerical results obtained by Keefe (1985) who considered the CGL equation with homogeneous Neumann boundary conditions and who always found two zero Lyapunov exponents.

4 Symmetric solutions

As mentioned in the previous section, it is the symmetries of the solutions which are important when computing Lyapunov exponents, rather than the symmetries of the equation itself. In this paper, we restrict attention to the reflectional symmetries s_1 and s_2 defined in Section 2 and consider perturbations which break the reflectional symmetries but have the same spatial period as the underlying solution. In a subsequent paper (Aston & Laing, 1999), we will consider finite subgroups of the space translation symmetry r_α—this corresponds to adding perturbations which have period greater than that of the underlying solution.

We define the two subgroups

$$\Sigma_1 = \{I, s_1\}, \qquad \Sigma_2 = \{I, s_1 s_2\}$$

Clearly, these two subgroups are both isomorphic to $\mathbb{Z}_2$ and their fixed point spaces are respectively the spaces of even and odd periodic functions. These are equivalent to homogeneous Neumann and homogeneous Dirichlet boundary conditions, respectively. A third symmetry which we could consider is the product of the previous two and is simply s_2. However, only the zero function is fixed by s_2 and so this is of no interest.

There is another group which is of interest where we combine one of the reflections with a space translation. This is given by

$$\Sigma_3 = \{I, s_1 s_2, r_\pi s_1, r_\pi s_2\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$$

We note that

$$r_\pi s_1 A(x,t) = A(\pi - x, t), \qquad r_\pi s_2 A(x,t) = -A(x+\pi, t)$$

Thus, functions fixed by $s_1 s_2$ are odd, functions fixed by $r_\pi s_1$ are even about $\pi/2$ and functions fixed by $r_\pi s_2$ satisfy $A(x+\pi,t) = -A(x,t)$. Clearly if two of these are satisfied then so is the third. Thus $\mathrm{Fix}(\Sigma_3)$ consists of functions which are odd (about zero) and even about $\pi/2$ which is equivalent to imposing a Dirichlet boundary condition at $x=0$ and a Neumann boundary condition at $x=\pi/2$. We note that a translation in space of a function in $\mathrm{Fix}(\Sigma_3)$ by $\pi/2$ gives a function which is even about $x=0$ and odd about $x=\pi/2$. Thus, interchanging the boundary conditions gives rise to a conjugate solution.

We are interested in determining the stability of chaotic solutions which have some symmetry to symmetry-breaking perturbations. For the first two cases of the subgroups Σ_1 and Σ_2, we decompose the space X as

$$X = X_e \oplus X_o$$

where X_e consists of all even functions of period 2π and X_o consists of all odd functions of period 2π. This is the isotypic decomposition associated with both the groups Σ_1 and Σ_2.

For solutions with symmetry Σ_1, we require solutions in $X_e = \mathrm{Fix}(\Sigma_1)$ and we compute the dominant Lyapunov exponent associated with perturbations in X_o. Similarly, for Σ_2, we compute solutions in $X_o = \mathrm{Fix}(\Sigma_2)$ and calculate the dominant Lyapunov exponent associated with perturbations in X_e. We note that Keefe (1989) computed the dominant Lyapunov exponent associated with the attractor in X_e and in X but did not look at the effect of symmetry-breaking perturbations.

The group Σ_3 has four one-dimensional irreducible representations and so there are four corresponding isotypic components. These can be specified as

$$W_1 = \{A \in X : A(0,t) = 0,\ A_x(\pi/2,t) = 0\} = \mathrm{Fix}(\Sigma_3)$$

$$W_2 = \{A \in X : A_x(0,t) = 0,\ A(\pi/2,t) = 0\}$$

$$W_3 = \{A \in X : A(0,t) = 0,\ A(\pi/2,t) = 0\}$$

$$W_4 = \{A \in X : A_x(0,t) = 0,\ A_x(\pi/2,t) = 0\}$$

and correspond to all possible combinations of homogeneous Neumann and Dirichlet boundary conditions at $x=0$ and $\pi/2$. In this case, there are again two zero Lyapunov exponents associated with W_1. Now if $A(x,t) \in \mathrm{Fix}(\Sigma_3)$, then $L_\alpha A(x,t) = A_x(x,t) \in W_2$ and so the third zero Lyapunov exponent is associated with W_2.

Expanding $A(x,t)$ as a Fourier series, it is easy to identify the modes which must occur for each isotypic component as

$$A(x,t) \in W_1 \Rightarrow A(x,t) = \sum_{k=1}^{\infty} b_k(t)\sin(2k-1)x + \mathrm{i}\left(\sum_{k=1}^{\infty} c_k(t)\sin(2k-1)x\right)$$

$$A(x,t) \in W_2 \Rightarrow A(x,t) = \sum_{k=1}^{\infty} b_k(t)\cos(2k-1)x + \mathrm{i}\left(\sum_{k=1}^{\infty} c_k(t)\cos(2k-1)x\right)$$

$$A(x,t) \in W_3 \Rightarrow A(x,t) = \sum_{k=1}^{\infty} b_k(t) \sin 2kx + \mathrm{i}\left(\sum_{k=1}^{\infty} c_k(t) \sin 2kx\right)$$

$$A(x,t) \in W_4 \Rightarrow A(x,t) = \frac{b_0(t)}{2} + \sum_{k=1}^{\infty} b_k(t) \cos 2kx + \mathrm{i}\left(\frac{c_0(t)}{2} + \sum_{k=1}^{\infty} c_k(t) \cos 2kx\right)$$

5 Numerical method

In this section we briefly describe the numerical method used for calculating the solutions shown in Section 6. We use a pseudo-spectral method coded in Matlab with time integration performed by a variable step-size Runge–Kutta method. We show details for only the real part of $A(x,t)$; the imaginary part is dealt with similarly. We write the real part of $A(x,t)$ at the points $\{x_n\}$ as

$$A_r(x_n,t) = \frac{1}{N}\sum_{k=0}^{N-1} X_k(t) \exp(\mathrm{i}kx_n), \qquad 1 \leq n \leq N \tag{9}$$

where

$$x_n = \left(\frac{n-1}{N}\right) 2\pi$$

(This is just the inverse discrete Fourier transform of $\{X_k(t)\}$.) The spectral coefficients, $X_k(t)$, are obtained from $\{A_r(x_n,t)\}$ via the discrete Fourier transform

$$X_k(t) = \sum_{n=1}^{N} A_r(x_n,t) \exp(-\mathrm{i}kx_n), \qquad 0 \leq k \leq N-1 \tag{10}$$

Since $A_r(x_n,t)$ is real

$$X_k(t) = \overline{X_{N-k}(t)}, \qquad k = 1, \ldots, N/2-1$$

and both $X_0(t)$ and $X_{N/2}(t)$ are real. Furthermore, if $A_r(x_n,t)$ is even then $\{X_k(t)\}$ is real, and if $A(x_n,t)$ is odd then $\{X_k(t)\}$ is purely imaginary.

We use $\{X_k(t)\}$ as our dependent variables, i.e. given $\{X_k(t)\}$ we want to know $\{\dot{X}_k(t)\}$. Using the spectral representation of our solution, (10), evaluation of the linear terms $RA + (1+\mathrm{i}\nu)\nabla^2 A$ in (1) is simple, and unlike a finite-difference scheme, the spatial differentiation is exact. To calculate the non-linear term $(1+\mathrm{i}\mu)A|A|^2$ the inverse transform (9) is used to form $\{A(x_n,t)\}$ from its real and imaginary parts. The cubic term $\{A(x_n,t)|A(x_n,t)|^2\}$ is then calculated, and a Fourier transform of the form (10) is used to evaluate the contribution of this non-linear term to $\{\dot{X}_k(t)\}$. Anti-aliasing is performed in the calculation of the non-linear term using padding and truncation as described in Canuto *et al.* (1988) and N is chosen to be a power of 2 so that the fast Fourier transform and its inverse can be used.

6 Numerical results

In Fig. 1 we show an example of a periodic orbit that is stable within X_e but which is unstable to odd perturbations. We start from a randomly chosen even initial

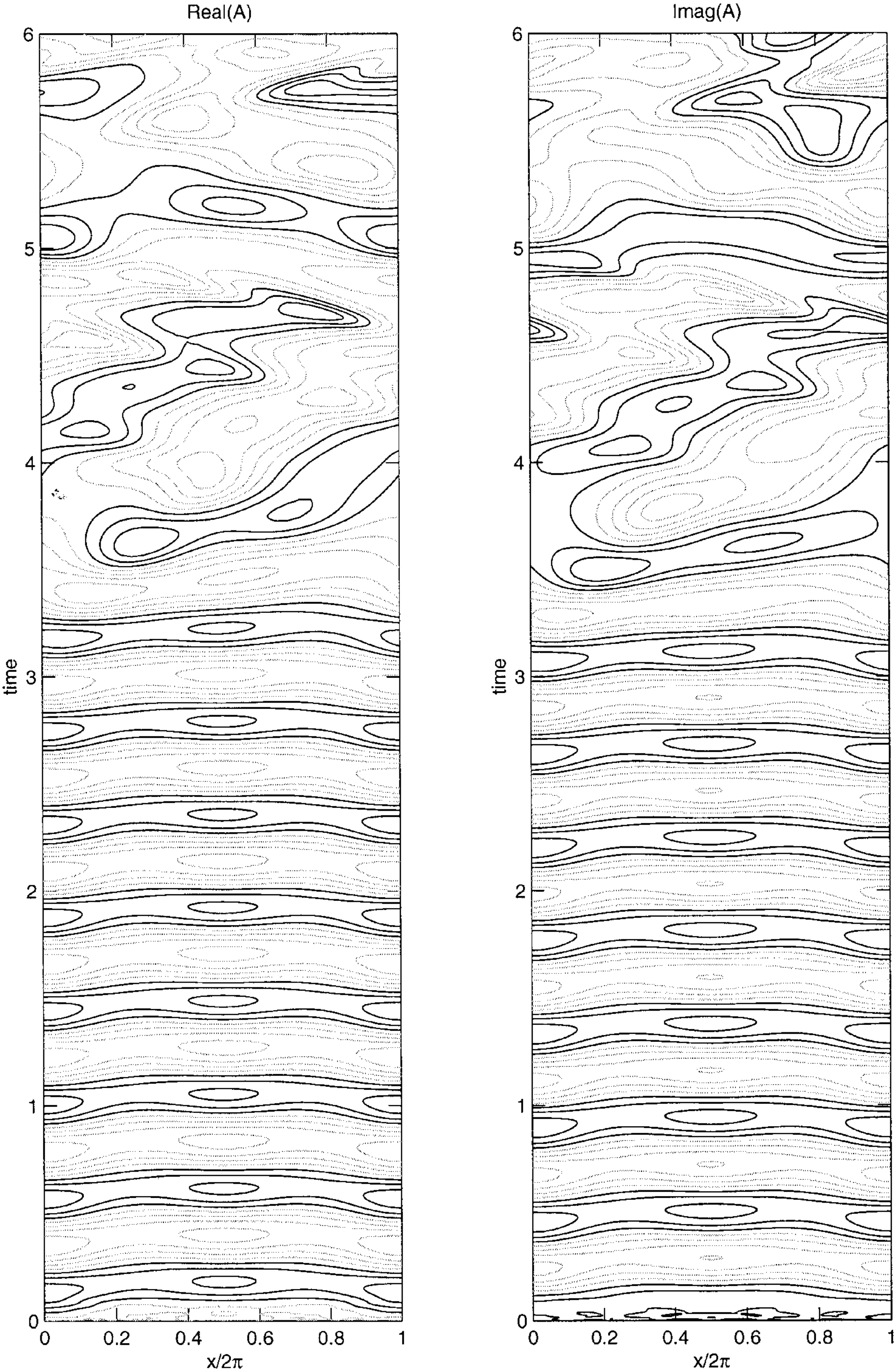

Fig. 1. A periodic even solution that is unstable to odd perturbations, leading to non-symmetric chaos. Parameter values: $R = 16$, $\mu = 1$, $\nu = -7$. A small odd perturbation is introduced at $t = 2$.

condition which evolves rapidly to a periodic state. A small odd perturbation is introduced at $t = 2$ and this grows exponentially in time so that for $t > 4$ the solution is chaotic with a spatial wavelength of 2π. Parameter values are $R = 16$, $\mu = 1$, $\nu = -7$. The real part of A is shown on the left and the imaginary part on the right. In Figs 1–4, 8 and 9 we show contour plots of the real and imaginary parts of the solution with black contour lines for negative values and grey contour lines for positive values. The x-axis has been rescaled to $[0, 1)$ in each of these figures.

Figure 2 shows an example of a chaotic solution in X_e that is unstable to odd perturbations. The randomly chosen even initial condition evolves to an even chaotic attractor. A small odd perturbation is introduced at $t = 2$ and this grows to produce a chaotic solution with a spatial wavelength of 2π. Parameter values are $R = 16$, $\mu = 5$, $\nu = -7$. Clearly there are two coexisting chaotic 'attractors' at these parameter values, although the even attractor is only attracting in the even subspace.

Figure 3 shows a chaotic odd solution that is unstable to even perturbations. A small even perturbation is added at $t = 0.5$ and this grows to produce a chaotic solution with no symmetry. Parameter values are $R = 36$, $\nu = -10$, $\mu = 12$.

Figure 4 shows a chaotic solution in $\mathrm{Fix}(\Sigma_3)$ that is unstable to perturbations in all three isotypic components, W_2, W_3, W_4. A small perturbation not in $\mathrm{Fix}(\Sigma_3)$ is added at $t = 1$ and this grows exponentially in time. Parameter values are $R = 16$, $\nu = -5$, $\mu = 12$.

In Fig. 5 we show the dominant Lyapunov exponents for a solution in $\mathrm{Fix}(\Sigma_3)$ as a function of $\mu (= -\nu)$ for $R = 16$. The continuous symmetries mentioned above prevent the dominant Lyapunov exponent associated with W_1 and W_2 from being negative but put no restrictions on the dominant Lyapunov exponents associated with W_3 and W_4. Note that the dominant Lyapunov exponent associated with W_1 is always zero and so the non-trivial Lyapunov exponents will all be negative. This implies that the underlying solution is either periodic or quasiperiodic. To illustrate the stability or instability of the underlying motion with respect to the different perturbations, we define projections on to each of the Σ_3-isotypic components as

$$A_1(x, t) = \frac{1}{4}[A(x, t) - A(-x, t) - A(x - \pi, t) + A(\pi - x, t)]$$

$$A_2(x, t) = \frac{1}{4}[A(x, t) + A(-x, t) - A(x - \pi, t) - A(\pi - x, t)]$$

$$A_3(x, t) = \frac{1}{4}[A(x, t) - A(-x, t) + A(x - \pi, t) - A(\pi - x, t)]$$

$$A_4(x, t) = \frac{1}{4}[A(x, t) + A(-x, t) + A(x - \pi, t) + A(\pi - x, t)]$$

Note that $A(x, t) = A_1(x, t) + A_2(x, t) + A_3(x, t) + A_4(x, t)$. In Fig. 6 we choose the parameter values $R = 16$, $\mu = 2.9 = -\nu$. We start with an initial condition in $\mathrm{Fix}(\Sigma_3)$, let this evolve until $t = 1.5$ and then add a small perturbation in either W_2, W_3 or W_4. The norm of the projection of the solution on to the appropriate isotypic component is then plotted as a function of time, along with the norm of the solution projected on to $W_1 = \mathrm{Fix}(\Sigma_3)$ on a logarithmic scale.

We see that the underlying solution is unstable with respect to perturbations in W_2 and W_4 but is stable with respect to perturbations in W_3, which is consistent

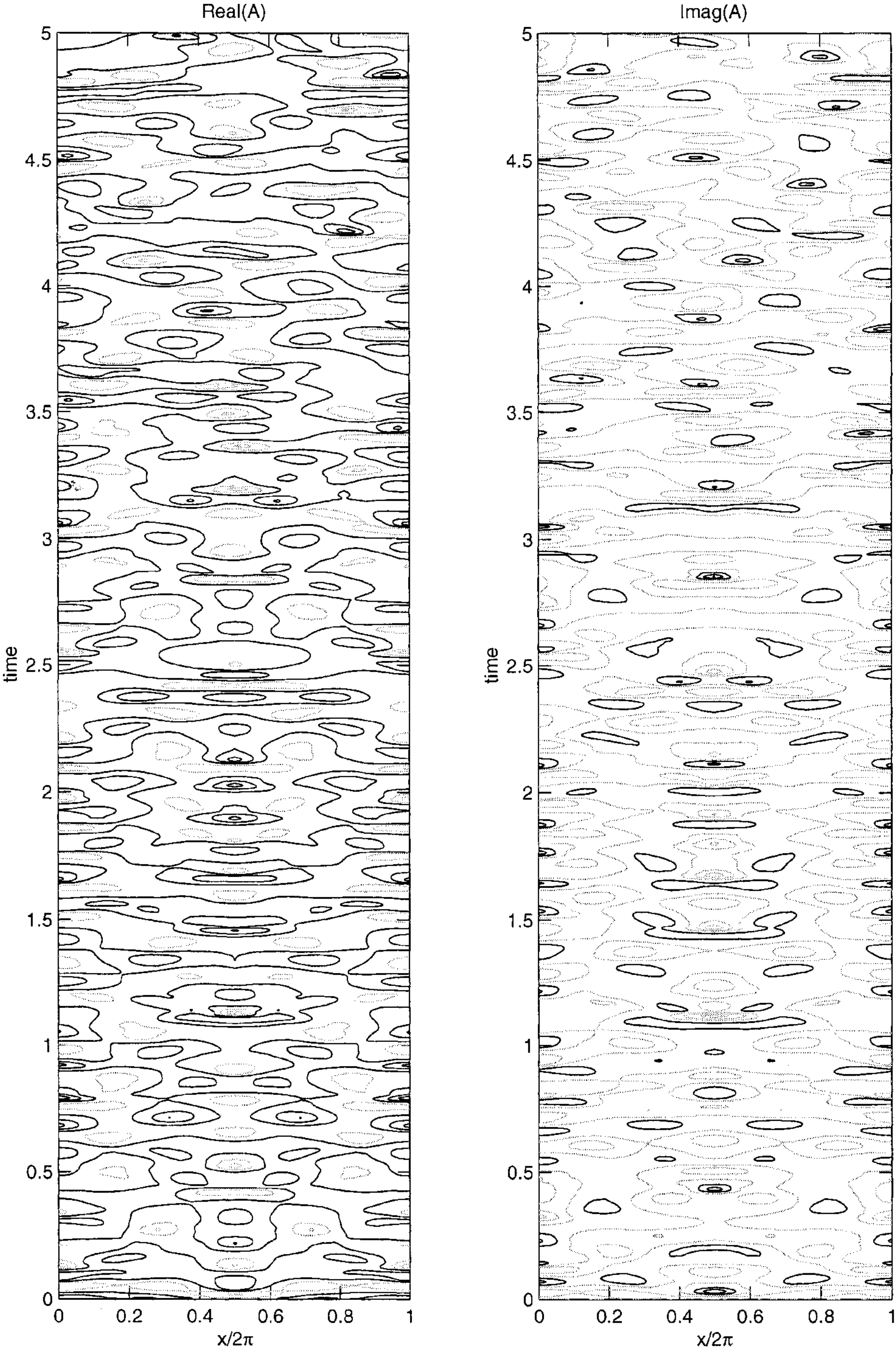

Fig. 2. A chaotic even solution that is unstable to odd perturbations, leading to non-symmetric chaos. Parameter values: $R = 16$, $\mu = 5$, $\nu = -7$. A small odd perturbation is introduced at $t = 2$.

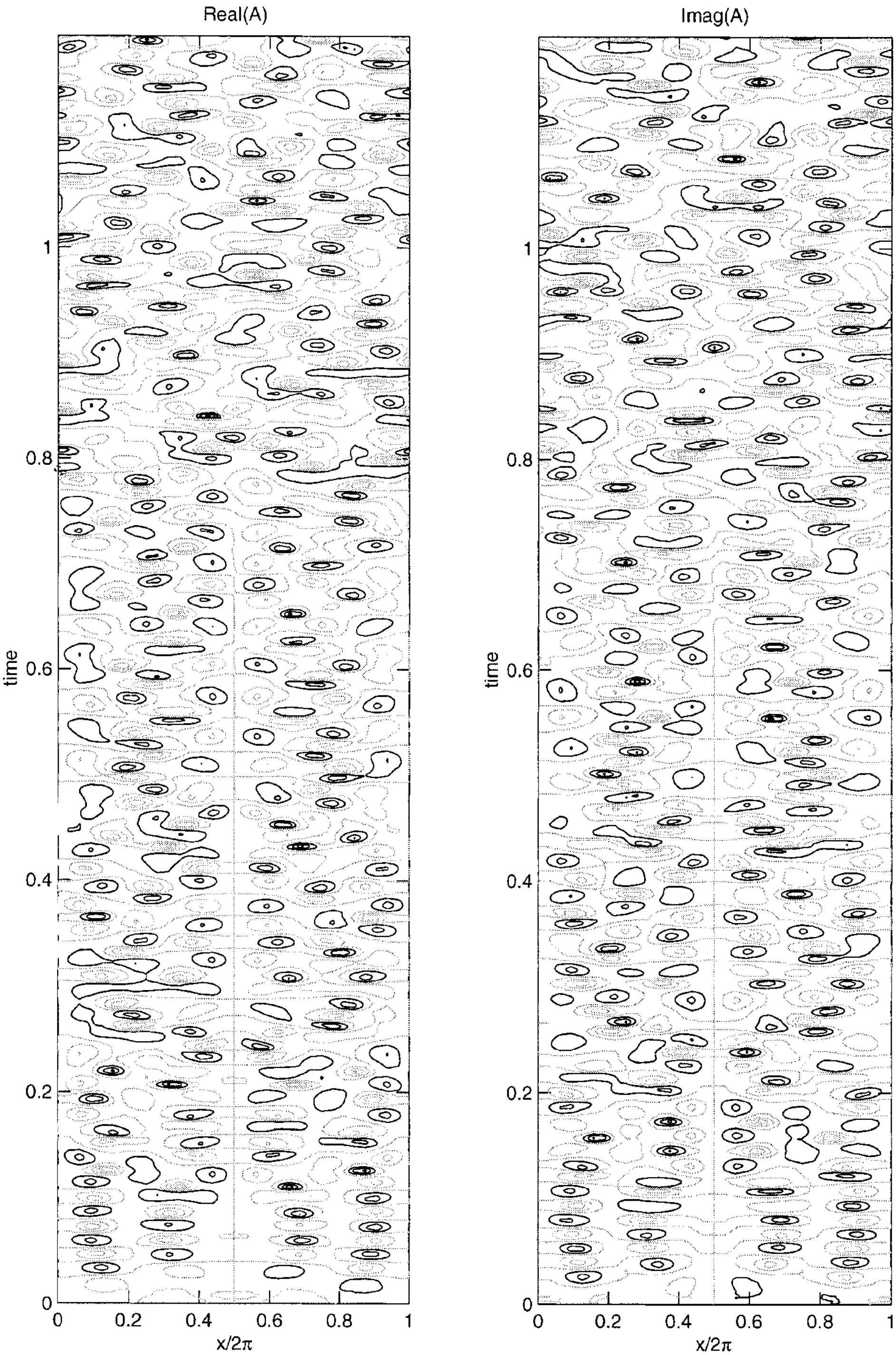

Fig. 3. A chaotic odd solution that is unstable to even perturbations. Parameter values: $R = 36$, $\nu = -10$, $\mu = 12$. A small even perturbation is added at $t = 0.5$.

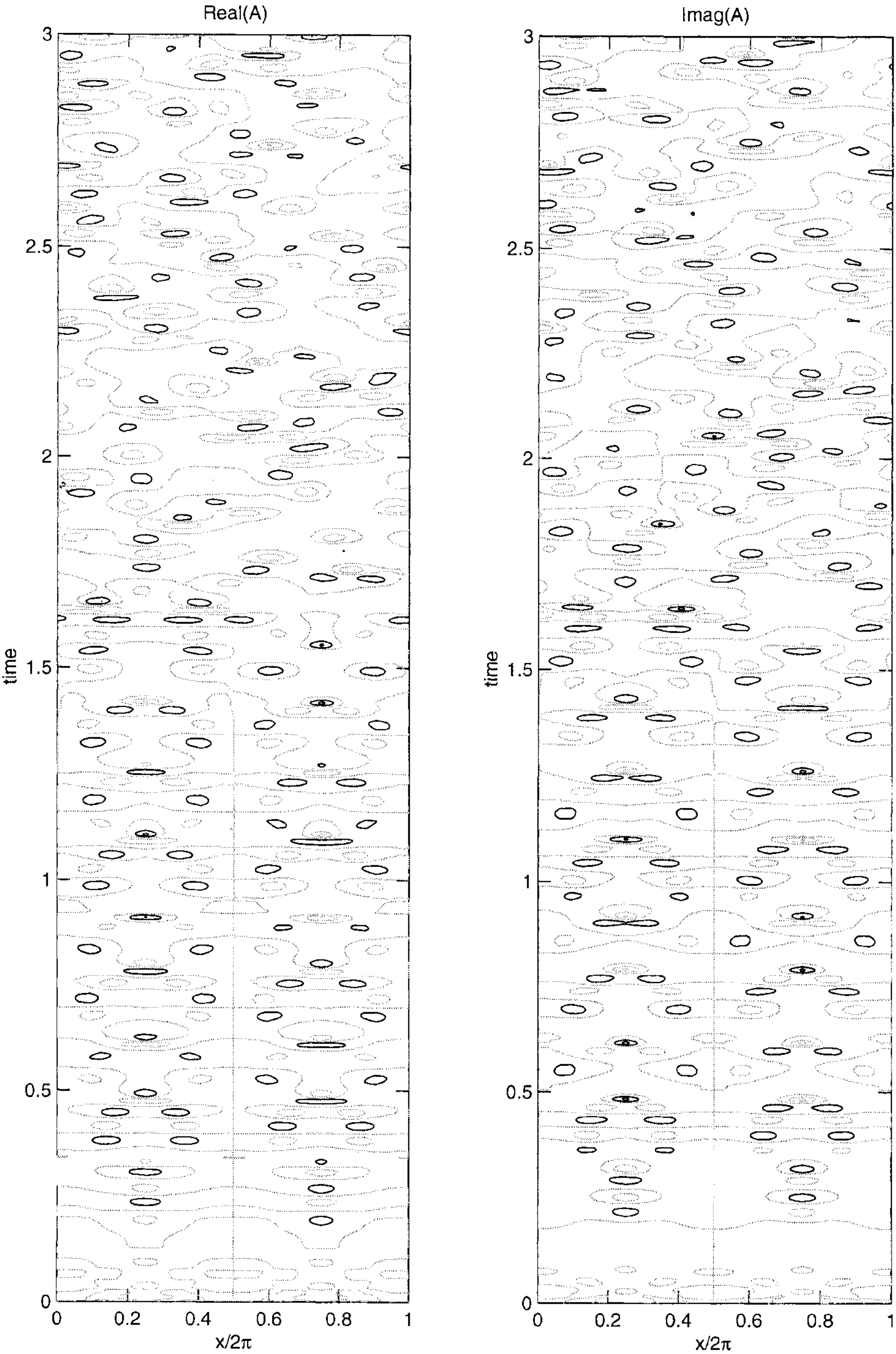

Fig. 4. A chaotic solution that is odd about 0 and even about $\pi/2$. Parameter values: $R = 16$, $\nu = -5$, $\mu = 12$. A small perturbation that is neither odd about 0 nor even about $\pi/2$ has been added at $t = 1$.

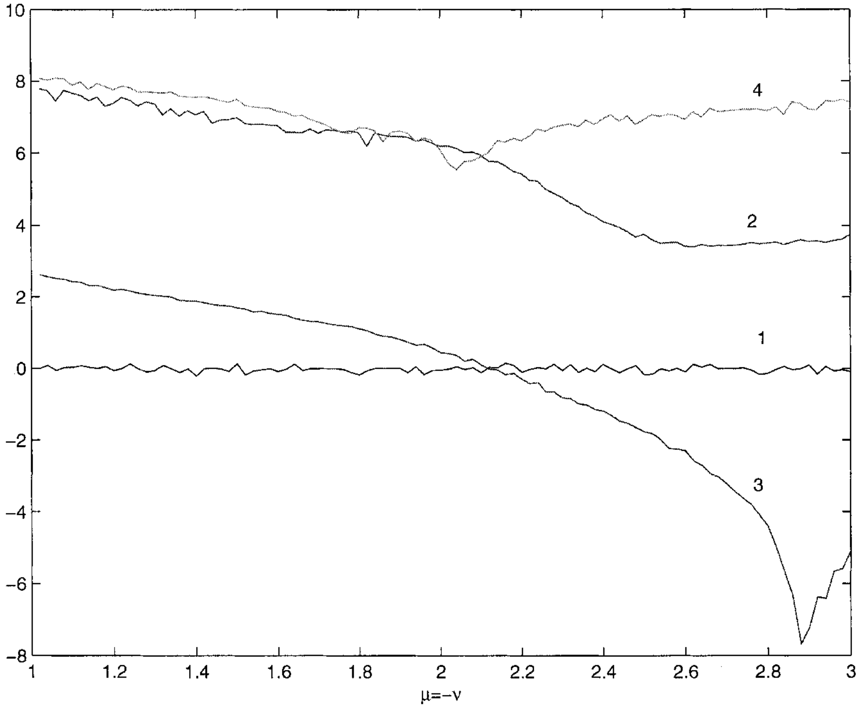

Fig. 5. Dominant Lyapunov exponents for a solution in Fix(Σ_3) with $R = 16$. 1: perturbations in W_1 (= Fix(Σ_3)), 2: perturbations in W_2, 3: perturbations in W_3, 4: perturbations in W_4.

with the signs of the corresponding Lyapunov exponents shown in Fig. 5. Note that the Lyapunov exponent associated with perturbations in W_4 is larger than for perturbations in W_2 and also that the solution component in W_1 ceases to be periodic when the W_4 component becomes similar in magnitude to it. This is the point at which, to the eye, the solution would no longer appear to have Σ_3 symmetry.

This type of instability has implications for the numerical computation of solutions with certain symmetries. The source of the problem is the numerical computation of the discrete Fourier transform and its inverse. Each of these processes introduces errors which are, for Matlab, typically smaller than the quantity being transformed by a factor of 10^{16}. Thus, for example, the discrete Fourier transform of a real, even function will not be purely real, but will have a small imaginary component, and when the inverse transform is taken the result will have a small odd component. If this even solution is stable with respect to odd perturbations this will not matter, but for solutions such as those shown in Figs 1 and 2 where we have instability to odd perturbations these errors will grow exponentially in time and ultimately overwhelm the even solution.

If we want the solution to remain within an invariant subspace we must modify the numerical scheme. For the above example, this is simply done by setting the imaginary component of the transformed variable to zero immediately after it is calculated, or better still, only working with the real part of the transformed variable. This is the technique used for Lyapunov exponent calculations. This method cannot, however, be used in, for example, the calculations for Fig. 1. Here,

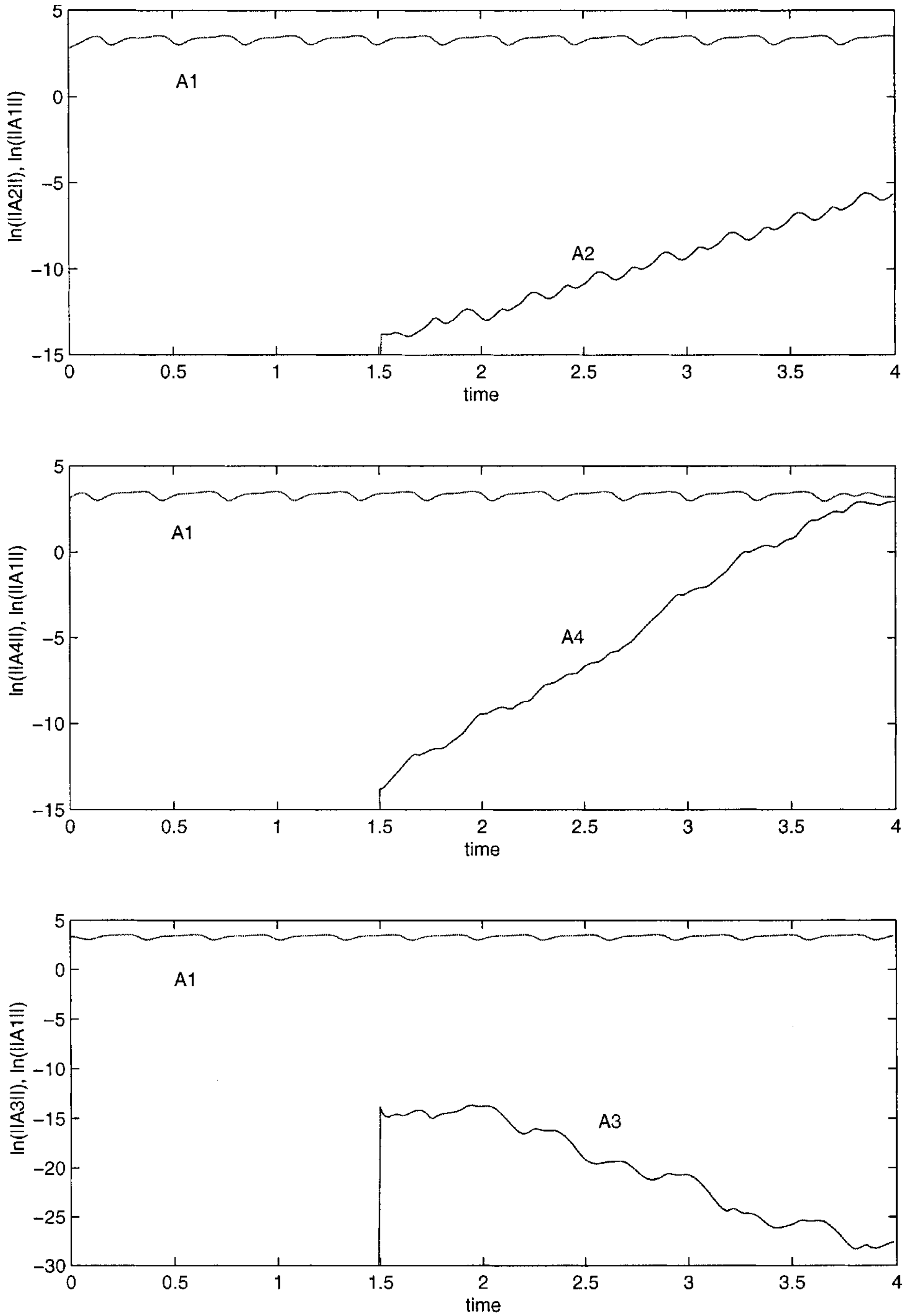

Fig. 6. Growth and decay of functions A_2, A_4 and A_3 over time with $R = 16$, $\mu = -\nu = 2.9$. In each case a small perturbation has been added at $t = 1.5$ to a solution in Fix(Σ_3).

although we added a small odd perturbation at time $t = 2$ s to demonstrate instability in this isotypic component, strictly speaking this was not necessary as, given sufficient time, the numerical errors introduced by the Fourier transform would have grown large enough to destroy the appearance of evenness—all we did was hasten the onset of this phenomenon.

6.1 Stable even chaos

Although for the vast majority of parameter values we examined, solutions that were chaotic when restricted to lie in a fixed point subspace were unstable with respect to perturbations normal to that space, we did find a region in parameter space in which there exist chaotic solutions in the subspace of even solutions that are stable with respect to perturbations in the odd subspace. We plot the Lyapunov exponents in part of this region in Fig. 7 and show a typical example, at parameter values $R = 1.05$, $\nu = 4$, $\mu = -4$, in Fig. 8, where this particular solution is even about the origin. We might describe this as 'weak' chaos arising from the bifurcation of a periodic or quasiperiodic orbit for which the dominant Lyapunov exponent in the normal direction remains zero as a parameter is varied.

The chaotic even solution at these parameter values is 'orbitally stable', meaning that there is a continuous family of such solutions related to one another by the spatial shift, r_α, and (assuming there are no coexisting attractors) an arbitrary initial condition will be attracted to one member of this family, i.e. a solution that is even about some point. We demonstrate this in Fig. 9 where we plot a solution started at a randomly chosen initial condition together with one of the two points in $[0, 2\pi)$ about which the first mode of the solution (i.e. that described by a linear

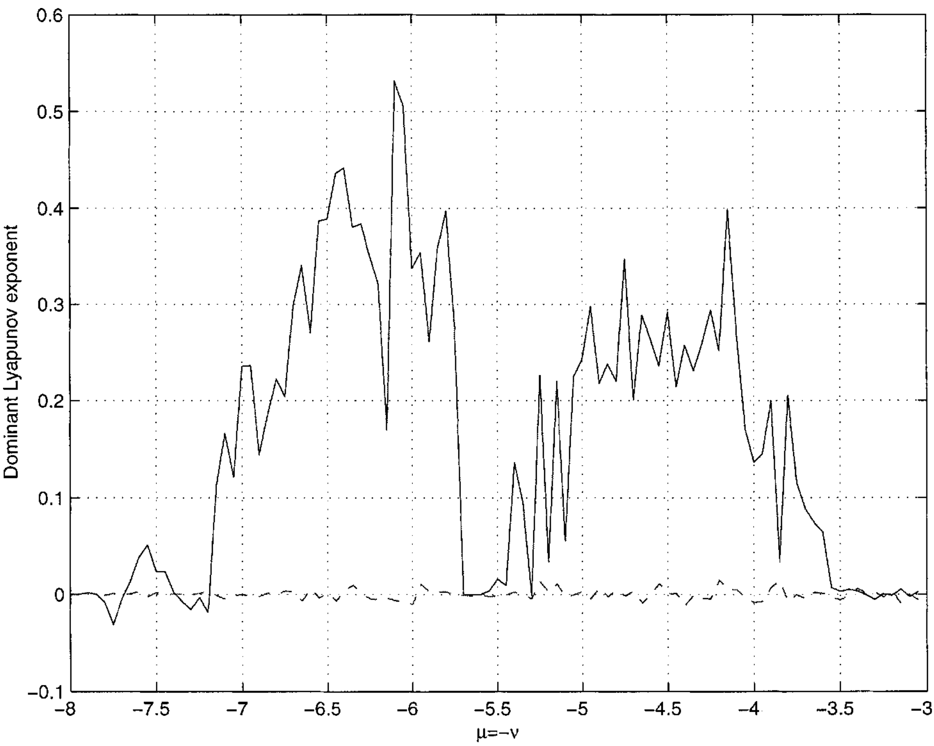

Fig. 7. Dominant Lyapunov exponents for an even solution (solid line) with an odd perturbation (dashed line) for $R = 1.05$.

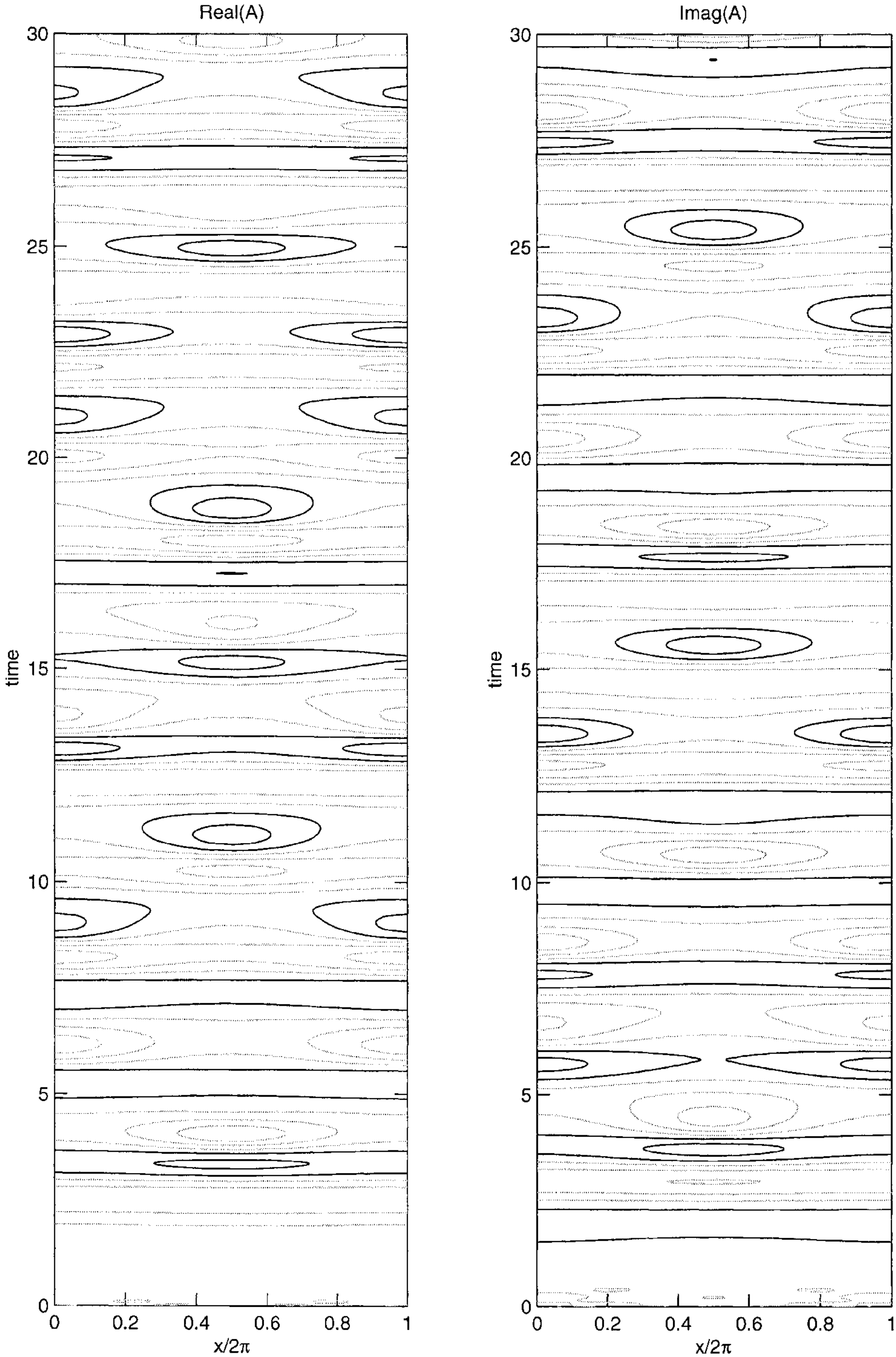

Fig. 8. A stable chaotic solution that is even about the origin. Parameter values: $R = 1.05$, $\nu = 4$, $\mu = -4$.

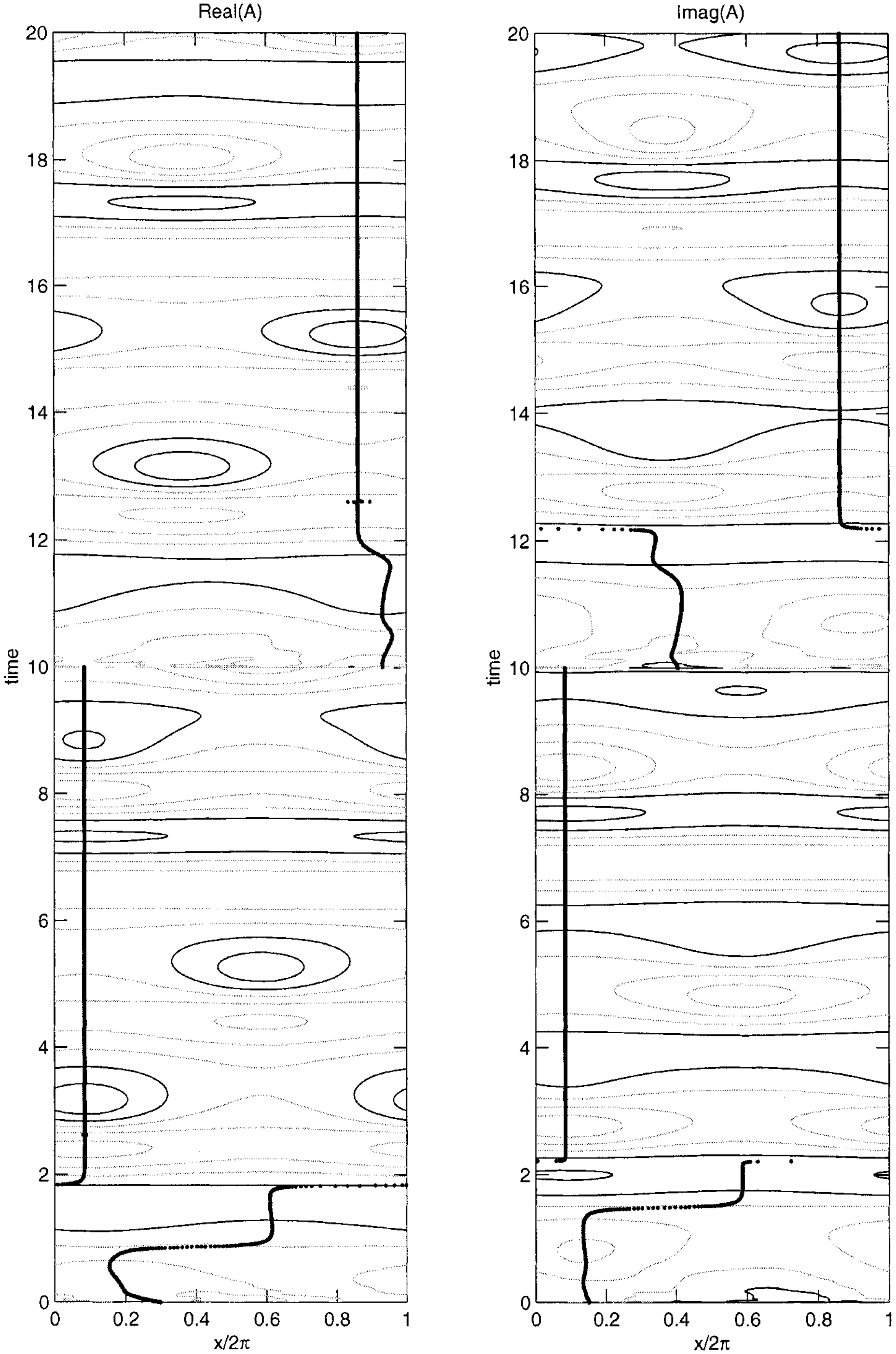

Fig. 9. Demonstration of the orbital stability of an even chaotic solution. The dark line is one of the two points about which the first mode is even, for both the real and imaginary parts of A. See text for more details. Parameter values: $R = 1.05$, $\nu = 4$, $\mu = -4$.

combination of $\sin x$ and $\cos x$) is even, for both the real and imaginary parts of A. We see that by time $t = 3$ s the solution has settled to being even about an x value of approximately $0.1 \times 2\pi$. A large randomly chosen perturbation was added at $t = 10$ and we see that the solution quickly settled to being even about an x value of approximately $0.87 \times 2\pi$. Note that during the transients the first modes of the real and imaginary parts of A are even about different points, but on the attractor they are even about the same point, as they must be for an even solution. See Golubitsky *et al.* (1988) for more details on orbital stability.

Despite searching, we could not find any evidence of a 'blowout' bifurcation (Ashwin *et al.*, 1998) in which the even solution remains chaotic while the dominant Lyapunov exponent in the normal direction changes from zero to positive as a parameter is varied. The reason for this is that, as shown in Fig. 7, the solution in the even subspace becomes periodic or quasiperiodic before the normal Lyapunov exponent becomes positive.

7 Conclusions

Our numerical results show that for much of the parameter space for the CGL equation, chaotic solutions which have some sort of reflectional symmetry are unstable to perturbations which break that symmetry, while there are also small regions of parameter space in which there are chaotic even solutions that are asymptotically stable with respect to odd perturbations. Of course we have not investigated all of the three-dimensional parameter space and there may be more interesting behaviour waiting to be found. Periodic solutions with symmetry are sometimes stable with respect to symmetry-breaking perturbations but it would appear that for most parameter values for the CGL equation that for arbitrary initial conditions, if the final solution is chaotic then it will have the minimum possible amount of symmetry. Clearly these ideas apply to any PDEs with symmetry and are not restricted to the CGL equation. Different results may be obtained for different equations.

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References

Alexander, J. C., Kan, I., Yorke, J. A. and You, Z. (1992) Riddled basins. *International Journal of Bifurcation & Chaos* **2**, 795–813.

Ashwin, P., Buescu, J. and Stewart, I. (1994) Bubbling of attractors and synchronisation of chaotic oscillators. *Physics Letters A* **193**, 126–139.

Ashwin, P., Buescu, J. and Stewart, I. (1996) From attractor to chaotic saddle: a tale of transverse instability. *Nonlinearity* **9**, 703–737.

Ashwin, P., Aston, P. J. and Nicol, M. (1998) On the unfolding of a blowout bifurcation. *Physica D* **111**, 81–95.

Aston, P. J. (1991) Analysis and computation of symmetry-breaking bifurcation and scaling laws using group theoretic methods. *SIAM Journal of Mathematical Analysis* **22**, 181–212.

Aston, P. J. and Dellnitz, M. (1995) Symmetry breaking bifurcations of chaotic attractors. *International Journal of Bifurcation & Chaos* **5**, 1643–1676.

Aston, P. J. and Laing, C. R. (1999) Symmetry and chaos in the complex Ginsburg–Landau equation—II. Translational symmetries. *Physica D* (in press).

Bartuccelli, M. V., Constantin, P., Doering, C. R., Gibbon, J. D. and Gisselfalt, M. (1990) On the possibility of soft and hard turbulence in the complex Ginsburg–Landau equation. *Physica D* **44**, 421–444.

Bartuccelli, M. V., Gibbon, J. D. and Oliver, M. (1996) Length scales in solutions of the complex Ginzburg–Landau equation. *Physica D* **89**, 267–286.

Canuto, C., Hussaini, M. Y., Quarteroni, A. and Zang, T. A. (1988) *Spectral Methods in Fluid Dynamics* (Springer-Verlag, Berlin).

Constantin, P. (1989) A construction of inertial manifolds. *Contemp. Math* 99 (Proceedings AMS Summer School, Boulder, 1987), pp. 27–62.

Covas, E., Ashwin, P. and Tavakol, R. (1997) Non-normal parameter blowout bifurcation in a truncated mean field dynamo. *Physical Review E* **56**, 6451–6458.

Covas, E., Tavakol, R., Ashwin, P., Tworkowski, A. and Brooke, J. M. (1998) In–out intermittency in PDE and ODE models of axisymmetric mean-field dynamos. Technical Report, Department of Mathematics and Statistics, University of Surrey.

Doering, C. R., Gibbon, J. D., Holm, D. D. and Nicolaenko, B. (1987) Exact Lyapunov dimension of the universal attractor for the complex Ginsburg–Landau equation. *Physical Review Letters* **59**, 2911–2914.

Doering, C. R., Gibbon, J. D., Holm, D. D. and Nicolaenko, B. (1988) Low-dimensional behaviour in the complex Ginzburg–Landau equation. *Nonlinearity* **1**, 279–309.

Doering, C. R., Gibbon, J. D. and Levermore, C. D. (1994) Weak and strong solutions of the complex Ginsburg–Landau equation. *Physica D* **71**, 285–318.

Duan, J. Q., Titi, E. S. and Holmes, P. (1993) Regularity, approximation and asymptotic dynamics for a generalized Ginzburg–Landau equation. *Nonlinearity* **6**, 915–933.

Eckmann, J. P. and Ruelle, D. (1985) Ergodic theory of chaos and strange attractors. *Reviews of Modern Physics* **57**, 617–656.

Golubitsky, M., Stewart, I. and Schaeffer, D. G. (1988) *Singularities and Groups in Bifurcation Theory*, Vol. II (Applied Math. Sciences 69, Springer-Verlag, Berlin).

Healey, T. (1989) Numerical bifurcation with symmetry: diagnosis & computation of singular points. *Bifurcation Theory and its Numerical Analysis* (eds Li Kaitai, J. E. Marsden, M. Golubitsky and G. Iooss; Xi'an Jiatong University Press), pp. 218–227.

Keefe, L. R. (1985) Dynamics of perturbed wavetrain solutions to the Ginzburg–Landau equation. *Studies in Applied Mathematics* **73**, 91–153.

Keefe, L. (1989) Properties of Ginzburg–Landau attractors associated with their Lyapunov vectors and spectra. *Physics Letters A* **140**, 317–322.

Kuramoto, Y. and Koga, S. (1981) Turbulized rotating chemical waves. *Progress in Theoretical Physics Supplement* **66**, 1081–1085.

Newell, A. C. and Whitehead, J. A. (1969) Finite bandwidth, finite amplitude convection. *Journal of Fluid Mechanics* **38**, 279–303.

Ogorzalek, M. J. (1993) Taming chaos—Part I: Synchronisation. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications* **40**, 693–699.

Oseledec, V. I. (1968) A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems. *Trudy Moskovskogo Matematicheskogo Obshchestva* **19**, 197–231.

Ott, E. and Sommerer, J. (1994) Blowout bifurcations: the occurrence of riddled basins and on–off intermittency. *Physics Letters A* **188**, 39–47.

Pecora, L. M. and Carroll, T. C. (1990) Synchronisation in chaotic oscillators. *Physical Review Letters* **64**, 821–824.

Stewartson, K. and Stuart, J. T. (1971) A non-linear instability theory for a wave system in plane Poiseulle flow. *Journal of Fluid Mechanics* **48**, 529–545.

Stuart, J. T. and DiPrima, R. C. (1978) The Eckhaus and Benjamin–Feir resonance mechanisms. *Proceedings of the Royal Society London A* **362**, 27–41.

Werner, B. (1990) Eigenvalue problems with the symmetry of a group and bifurcations. *Continuation and Bifurcations: Numerical Techniques and Applications*, NATO ASI Series C, Vol. 313 (eds D. Roose, B. De Dier and A. Spence), pp. 71–88.

Yamada, H. and Fujisaka, T. (1983) Stability theory of synchronised motion in coupled-oscillator systems. *Progress in Theoretical Physics* **69**, 32–47.