ELSEVIER

12 February 1996

PHYSICS LETTERS A

Physics Letters A 211 (1996) 155–160

A homoclinic hierarchy

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Received 29 June 1995; revised manuscript received 13 November 1995; accepted for publication 30 November 1995
Communicated by A.P. Fordy

Abstract

Homoclinic bifurcations in autonomous ordinary differential equations provide useful organizing centres for the analysis of examples. There are four generic types of homoclinic bifurcation, depending on the dominant eigenvalues of the Jacobian matrix of the flow near a stationary point. A family of differential equations is presented which, for suitable choices of parameters, can exhibit each of these four homoclinic bifurcations. In one of the cases this provides the first smooth example of the bifurcation in the literature.

AMS classification: 58F13; 58F14

Keywords: Bifurcation; Homoclinic orbit; Chaos

A homoclinic orbit of an autonomous ordinary differential equation is a nontrivial solution, $x_H(t)$, which tends to a stationary point, x_0, in both forward and backward time, i.e. $x_H(t) \to x_0$ as $t \to \pm\infty$ and $x_H(0) \neq x_0$. In typical (e.g. non-Hamiltonian) families of ordinary differential equations the existence of a homoclinic orbit is not a structurally stable situation, and typical perturbations of the system will no longer have a homoclinic orbit close to the original one. Thus, in a one-parameter family of ordinary differential equations, there may be a parameter, $\mu = \mu_H$ say, at which the system has a homoclinic orbit and an $\epsilon > 0$ such that if $\mu \in (\mu_H - \epsilon, \mu_H + \epsilon)\backslash\{\mu_H\}$ there is no homoclinic orbit close to the orbit which exists at $\mu = \mu_H$. If this is the case we say that there is a homoclinic bifurcation at $\mu = \mu_H$.

The study of homoclinic bifurcations goes back (at least) to Poincaré, and later Andronov. More recent work has been stimulated by a series of papers by Shilnikov [1–3] in which it was shown that, given certain conditions described below, there is chaotic motion in a tubular neighbourhood of the homoclinic orbit, although the net effect of the bifurcation is to create a single periodic orbit (see, e.g., Ref. [4] for a discussion). Complicated sequences of local bifurcations at parameter values near μ_H may also occur [5,4] as well as more complicated homoclinic bifurcations, for which the homoclinic orbit loops several times through the tubular neighbourhood of the original homoclinic orbit [6].

These theoretical results can be a great help when investigating examples. The conditions which determine whether complicated dynamics occurs are based

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0375-9601/96/$12.00
SSDI 0375-9601(95)00953-1

on the linearized flow near the stationary point. Suppose that the stationary point is hyperbolic. Then, after a change of coordinates we may assume that it is at the origin for all values of μ which are of interest and the family of differential equations can be written in the form

$$\dot{x} = Ax + F(x, \mu) \tag{1}$$

for $x \in \mathbb{R}^n$, $n \geqslant 2$. Here $F(0, \mu) = 0$, A is a constant $n \times n$ matrix and F is smooth and contains only nonlinear terms. Assume that if $\mu = 0$ then the system has a homoclinic orbit, $x_H(t)$, biasymptotic to the origin, and that if $\mu \in (-\epsilon, \epsilon) \setminus \{0\}$ there are no homoclinic orbits close to x_H (by close we mean that for η sufficiently small $|x(t) - x_H(t)| < \eta$ for all $t \in (-\infty, \infty)$).

Now, since $x = 0$ is hyperbolic, the eigenvalues of A can be divided into two sets, $\{\lambda_i\}$, $i = 1, \dots, n_u$, and $\{\nu_i\}$, $i = 1, \dots, n_s$, $n_s + n_u = n$, such that $\mathrm{Re}(\lambda_i) > 0$ and $\mathrm{Re}(\nu_i) < 0$. These can be ordered so that

$$\begin{aligned} \mathrm{Re}(\nu_{n_s}) \leqslant \dots \leqslant \mathrm{Re}(\nu_2) \leqslant \mathrm{Re}(\nu_1) < 0 \\ < \mathrm{Re}(\lambda_1) \leqslant \mathrm{Re}(\lambda_2) \leqslant \dots \leqslant \mathrm{Re}(\lambda_{n_u}). \end{aligned} \tag{2}$$

Typically, trajectories which tend to $x = 0$ as $t \to \infty$ do so tangential to the eigenspace corresponding to those eigenvalues with $\mathrm{Re}(\nu_j) = \mathrm{Re}(\nu_1)$, which we refer to as the dominant stable eigenvalues. Similarly, almost all trajectories which tend to $x = 0$ as $t \to -\infty$ do so tangential to the eigenspace corresponding to the dominant unstable eigenvalues, i.e. those with $\mathrm{Re}(\lambda_j) = \mathrm{Re}(\lambda_1)$. We assume that the homoclinic orbit, $x_H(t)$ is typical in this sense.

There are four generic cases (up to time reversal).

(I) *Saddle-node homoclinic orbit.* The set of dominant eigenvalues is $\{\nu_1, \lambda_1\}$, with $\nu_1, \lambda_1 \in \mathbb{R}$, and $\nu_1 + \lambda_1 \neq 0$.

In this case (which can occur for $n \geqslant 2$), provide some genericity conditions are satisfied, the homoclinic bifurcation creates a single periodic orbit which exists in either $\mu < 0$ or $\mu > 0$ [2]. As μ tends to zero from the appropriate side the periodic orbit tends to the homoclinic orbit and the period of the orbit tends to infinity as the logarithm of $|\mu|$ for typical parametrizations. If $n_u = 1$ then the orbit is stable if $\nu_1 + \lambda_1 < 0$, otherwise it is a saddle.

(II) *Saddle-focus homoclinic orbit.* The set of dominant eigenvalues is $\{\nu_2, \nu_1, \lambda_1\}$, with $\nu_1 = \nu_2^* \in \mathbb{C} \setminus \mathbb{R}$, $\lambda_1 \in \mathbb{R}$, and $\mathrm{Re}(\nu_1) + \lambda_1 \neq 0$.

This case can occur if $n \geqslant 3$. There are two subcases.

(IIa) $\mathrm{Re}(\nu_1) + \lambda_1 < 0$. The bifurcation is essentially the same as case (I).

(IIb) $\mathrm{Re}(\nu_1) + \lambda_1 > 0$. If $\mu = 0$ there are chaotic solutions in a tubular neighbourhood of the homoclinic orbit. There are sequences of saddle-node bifurcations accumulating on $\mu = 0$ from both sides, and sequences of (geometrically more complicated) homoclinic bifurcations accumulating on $\mu = 0$ from one side only [1,3,5–9].

(III) *Bifocal homoclinic orbit.* The set of dominant eigenvalues is $\{\nu_2, \nu_1, \lambda_1, \lambda_2\}$ with $\nu_1 = \nu_2^* \in \mathbb{C} \setminus \mathbb{R}$ and $\lambda_1 = \lambda_2^* \in \mathbb{C} \setminus \mathbb{R}$.

This case can arise if $n \geqslant 4$. The dynamics is similar to that described for case (IIb), but typically there are more complicated homoclinic bifurcations on both sides of the bifurcation point $\mu = 0$ [10,7,3].

The results sketched above form the basis of global bifurcation theory, analogous to statements about the saddle-node, period-doubling and Hopf bifurcations in local bifurcation theory. Whilst there are many examples of cases (I) and (II) in the literature it is extraordinary that (to the best of our knowledge) no unambiguous examples of case (III) have been described to date. There are examples with homoclinic orbits to stationary points satisfying the spectral condition of case (III), but these are nongeneric, being in Hamiltonian or reversible systems, which have a very special bifurcation structure [11,12]. A piecewise linear example of case III is described in Ref. [13], and here we use the same ideas, described below, to construct a smooth (polynomial) system for which there is strong numerical evidence for the existence of a bifocal homoclinic orbit. In so doing we derive a hierarchy of equations in two, then three, and then four dimensions. Each equation is obtained from the previous system by extending it in an appropriate manner to an extra dimension. In principle this construction could be extended to obtain a hierarchy of equations in higher and higher dimensions each having a homoclinic orbit to a stationary point with a prescribed spectrum.

Simple examples of interesting dynamical phenomena have been constructed using a variety of techniques. Arnéodo, Coullet and Tresser [14] used piecewise linear systems to prove the existence of a case (IIb) saddle-focus homoclinic orbit, whilst Deng [15] uses slow manifolds. Here (cf. Ref. [13]) we use the adjoint eigenvectors of the linear part of a "seed" equation to define the coupling between the equation and an extra variable in such a way that the linear part of the new equation has the desired spectral condition. We then appeal to perturbation theory and numerical experiment to suggest that the dynamically interesting behaviour (in this case, the existence of a homoclinic orbit) is inherited by the new equation from the "seed" equation. The new equation can in turn be treated as a "seed" equation and the process can be repeated. The use of adjoint eigenvectors is not entirely necessary (one could try trial and error) but ensures that complete control of the spectral properties of the stationary point is maintained throughout the hierarchy.

Two-dimensional examples illustrating case (I) are easy to find, so let

$$\dot{x} = Ax + f(x, \mu) \tag{3}$$

be one such example ($x \in \mathbb{R}^2$, f is a smooth function of the plane to itself which contains only nonlinear terms, $f(0, \mu) = 0$ and there is a homoclinic orbit, biasymptotic to the stationary point at the origin if $\mu = 0$). Assume that the eigenvalues of the constant 2×2 matrix A are ν_1 and λ_1 with $\nu_1 < 0 < \lambda_1$ and

$$|\nu_1| > \lambda_1. \tag{4}$$

Now let e_s and e_u be the eigenvalues of A corresponding to the eigenvalues ν_1 and λ_1 respectively, and let $e_s^\dagger$ and $e_u^\dagger$ be the corresponding adjoint eigenvectors (see e.g. Ref. [16] for a discussion of adjoint eigenvectors in dynamical systems). Thus $A^T e_s^\dagger = \nu_1 e_s^\dagger$, $A^T e_u^\dagger = \lambda_1 e_u^\dagger$, $e_s^\dagger \cdot e_u = e_u^\dagger \cdot e_s = 0$ and the eigenvectors can be normalized so that $e_s^\dagger \cdot e_s = e_u^\dagger \cdot e_u = 1$.

Eq. (3) is the first member of the homoclinic hierarchy. Now define the extended system

$$\dot{x} = Ax - z e_s + f(x, \mu),$$

$$\dot{z} = \epsilon_1 (e_s^\dagger \cdot x) + \nu_1 z. \tag{5}$$

In coordinates (x_u, x_s, z) defined by $x = x_s e_s + x_u e_u$ the linear part of this equation is obtained by dotting through with $e_u^\dagger$ and $e_s^\dagger$ to give

$$\dot{x}_u = \lambda_1 x_u, \quad \dot{x}_s = \nu_1 x_s - z, \quad \dot{z} = \epsilon_1 x_s + \nu_1 z, \tag{6}$$

with eigenvalues $\lambda_1 > 0$ and $\nu_1 \pm \sqrt{-\epsilon_1}$. Hence if $\epsilon_1 > 0$ the linear part of (3) satisfies the conditions of case (IIa). Since homoclinic bifurcations are typically of codimension one we expect (at least for small $\epsilon_1 > 0$) there to be a curve of homoclinic bifurcations in (μ, ϵ_1) parameter space of the form $\mu = H(\epsilon_1)$ with $H(0) = 0$. If this curve does exist then (5) provides an example of case (IIa).

Similarly, if we consider

$$\dot{w} = \epsilon_2 (e_u^\dagger \cdot x) + \lambda_1 w,$$

$$\dot{x} = Ax - w e_u + f(x, \mu), \tag{7}$$

the linear part of the equation has eigenvalues $\nu_1 < 0$ and $\lambda_1 \pm \sqrt{-\epsilon_2}$ and so, using (4), under similar assumptions we obtain homoclinic bifurcations of class (IIb) in reverse time if $\epsilon_2 > 0$.

Finally, putting Eqs. (5) and (7) together to obtain

$$\dot{w} = \epsilon_2 (e_u^\dagger \cdot x) + \lambda_1 w,$$

$$\dot{x} = Ax - z e_s - w e_u + f(x, \mu),$$

$$\dot{z} = \epsilon_1 (e_s^\dagger \cdot x) + \nu_1 z, \tag{8}$$

we should be able to find bifocal homoclinic bifurcations (case (III)) if ϵ_1 and ϵ_2 are small and positive, the eigenvalues of the linear flow at the origin being $\lambda_1 \pm i\sqrt{\epsilon_2}$ and $\nu_1 \pm i\sqrt{\epsilon_1}$.

To illustrate the hierarchy (3), (5), (7) and (8), we have chosen, rather arbitrarily, to start with the two-dimensional system

$$\dot{x} = y, \quad \dot{y} = 6x - y - 6x^2 + \mu xy, \tag{9}$$

for which there is strong numerical evidence that a homoclinic orbit exists if $\mu = \mu_H \approx 1.164371$. For this example, in the notation of (3),

$$A = \begin{pmatrix} 0 & 1 \\ 6 & -1 \end{pmatrix}, \quad f(x, \mu) = \begin{pmatrix} 0 \\ -6x^2 + \mu xy \end{pmatrix}, \tag{10}$$

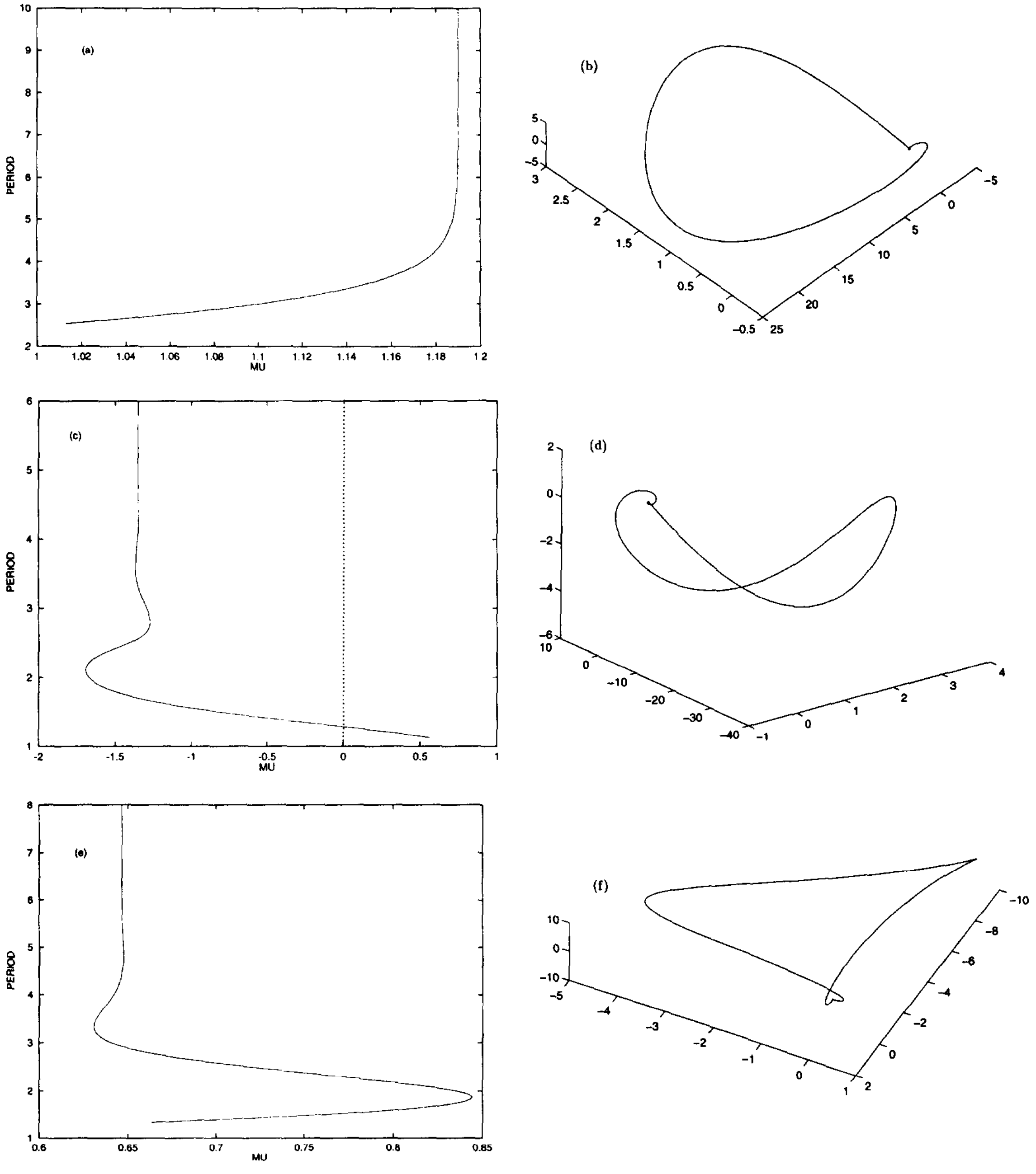

Fig. 1. Numerical simulations of (5), (7) and (8) with A, f and the adjoint eigenvectors given by (10) and (11). (a) Period against parameter (μ) for Eq. (5) with $\epsilon_1 = 0.1$ showing the approach of the simple periodic orbit to the homoclinic orbit. (b) A homoclinic orbit of (5) with $\epsilon_1 = 16$ and $\mu \approx 2.556795$. (c) As (a) using Eq. (7) with $\epsilon_2 = 16$. (d) A homoclinic orbit of (7) with $\epsilon_2 = 16$ and $\mu \approx -1.351357$. (e) As (a) using Eq. (8) (or, equivalently, (12)) with $\epsilon_1 = \epsilon_2 = 4.260467$. (f) A homoclinic orbit of (8) with $\epsilon_1 = \epsilon_2 = 4.260467$ and $\mu \approx 0.6466121$. In all cases the periodic orbits have been followed to much higher periods than plotted, and the "homoclinic orbits" are in fact periodic orbits of large period (greater than 100 in all three cases) which, we assume, are good approximations to the homoclinic orbits.

so $\lambda_1 = 2$, $\nu_1 = -3$ and (4) is satisfied. A simple calculation gives

$$e_u = \frac{1}{5}\begin{pmatrix}1\\2\end{pmatrix}, \quad e_u^\dagger = \begin{pmatrix}3\\1\end{pmatrix},$$
$$e_s = \frac{1}{5}\begin{pmatrix}1\\-3\end{pmatrix}, \quad e_s^\dagger = \begin{pmatrix}2\\-1\end{pmatrix}, \tag{11}$$

and so (8), from which (5) and (7) follow, becomes

$$\dot{w} = \epsilon_2(3x+y) + 2w, \quad \dot{x} = y - \tfrac{1}{5}z - \tfrac{1}{5}w,$$
$$\dot{y} = 6x - y + \tfrac{3}{5}z - \tfrac{2}{5}w - 6x^2 + \mu xy,$$
$$\dot{z} = \epsilon_1(2x-y) - 3z. \tag{12}$$

Although our argument for the existence of homoclinic orbits in (12) (and hence (5) and (7)) is essentially perturbative ($|\epsilon_1|$ and $|\epsilon_2|$ small), numerical experiments suggest that the curve of homoclinic orbits exists over a broad range of values of $|\epsilon_i|$ ($i = 1, 2$). We use larger values of the parameters to illustrate our results since the qualitative features of the orbits, in particular the spiralling motion near the stationary point, is much clearer at these values. In all cases, the approximate parameter value of the homoclinic bifurcation is obtained by following a periodic orbit using AUTO [17] to very high period with changing parameter. The homoclinic orbit can be thought of as the limit of this orbit as the period tends to infinity.

Fig. 1 shows the results of three sets of numerical experiments obtained using AUTO [17]. In Figs. 1a, 1b we have set $\epsilon_2 = w = 0$ (equivalent to choosing (5) with A and f given by (10) and the adjoint eigenvectors by (11)). This figure shows a plot of the period of a simple periodic orbit as a function of μ illustrating the familiar logarithmic increase in period as the orbit approaches the homoclinic orbit in case (IIa) with $\epsilon_1 = 0.1$. In Fig. 1b we show a homoclinic orbit for this system with $\epsilon_1 = 16$ and $\mu \approx 2.556795$, again corresponding to case (IIa).

Figs. 1c, 1d shows similar plots for $\epsilon_1 = z = 0$ and $\epsilon_2 = 16$ (equivalent to (7): $z = 0$ is an invariant manifold). In this case, as expected for (IIb), the periodic orbit undergoes a sequence of saddle-node bifurcations as its period tends to infinity. The homoclinic orbit at $\mu \approx -1.351357$ is illustrated in Fig. 1d.

Finally, Figs. 1e, 1f show the analogous pictures for $\epsilon_1 = \epsilon_2 = 4.260467$, illustrating the approach of the periodic orbit to a bifocal homoclinic orbit, which exists for $\mu \approx 0.6466121$. Fig. 1f does not show the homoclinic orbit, but an orbit of extremely long period (around 200) which lies close to the homoclinic orbit.

We consider that the fact that it is possible to follow a periodic orbit to very high period provides very strong evidence for the existence of the homoclinic orbit, but we have also done further numerical experiments to add more weight to our claim. The local stable manifold of the origin is tangential to the plane spanned by $e_1 = (0, 0, 0, 1)^T$ and e_s (extended to $\mathbb{R}^4$ in the obvious way) whilst the local unstable manifold is tangential to the plane spanned by e_u (extended to $\mathbb{R}^4$) and $e_4 = (1, 0, 0, 0)^T$. If a homoclinic orbit exists for the system then the stable and unstable manifolds intersect, and the numerically computed approximation shown in Fig. 1f suggests that a point of intersection lies in the hyperplane $y = 0$ with $2 < x < 2.5$. To demonstrate the existence of this intersection we integrated points on a circle of initial conditions enclosing the origin on the linear approximation to the local unstable manifold forwards in time and monitored the first intersection of these trajectories with the hyperplane $y = 0$ with $2 < x < 2.5$ (if such an intersection exists). In this way we obtained a series of points on a curved line segment, U. A similar exercise in reverse time using initial conditions on the linear approximation to the local stable manifold provided a second curved line segment, S. This numerical experiment was repeated at different values of μ. Using polynomial interpolation to obtain approximations for U and S between the computed points, the shortest vector from U to S was calculated using Newton's method on the parametrized curves. Now let n be the vector obtained in this way with $\mu = 0.64$, and $u(\mu)$ the vector obtained at nearby values of μ. These results allow us to form the signed distance function $\text{sign}(n \cdot u(\mu))|u(\mu)|$.

A zero of this signed distance function thus indicates an intersection between S and U, and hence the existence of a homoclinic orbit. If, in addition, the sign of the signed distance function changes, then the family of differential equations parametrized by μ passes transversely through the codimension one surface of systems with homoclinic orbits. We found, using a circle of radius 10^{-4} for the initial conditions and numerically obtained normalized eigenvectors, that for $0.55 < \mu < 0.64$ the signed distance function is positive (and equal to 0.004617 at $\mu = 0.64$ whilst for $0.65 < \mu < 0.71$ the signed distance function is negative (and equal to -0.002365 at $\mu = 0.65$). This strongly suggests that for some values of μ between 0.64 and 0.65 there is a zero of the distance function, and hence a homoclinic orbit for the differential equation (12). Linear interpolation between $\mu = 0.64$ and $\mu = 0.65$ gives an approximate value of $\mu = 0.6466$ for the homoclinic bifurcation, in excellent agreement with the value obtained by following periodic orbits.

We have written down a hierarchy of differential equations which illustrate the four fundamental homoclinic bifurcations. In particular, we have obtained a smooth example of a bifocal homoclinic bifurcation (case (III)). So far as we are aware, this is the first such example (in Ref. [13] a piecewise linear example is studied, for which the existence of a bifocal homoclinic bifurcation can be proved using perturbation theory, but this does not satisfy the standard smoothness conditions of Shilnikov's results [1,3] although the results can be trivially extended to such systems; the examples of Refs. [11,12] are non-generic, having either a Hamiltonian or reversible structure).

The observant reader will have noted that one way of interpreting example (12) is through the unfolding of the degenerate Jordan normal form

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \nu_1 & 1 \\ 0 & 0 & 0 & \nu_1 \end{pmatrix} \tag{13}$$

We look at the existence of bifocal homoclinic orbits in this light elsewhere [8]: in particular, we explore several codimension two bifurcations involving bifocal homoclinic bifurcations. The normal form (13) has codimension greater than two, and we consider this to be too large for useful analysis in the absence of some concrete physical motivation.

C.L. is grateful to the Cambridge Commonwealth Trust for financial support.

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