Chimeras in networks with purely local coupling

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Chimera states in spatially extended networks of oscillators have some oscillators synchronised while the remainder are asynchronous. These states have primarily been studied in networks with nonlocal coupling, and more recently in networks with global coupling. Here we present three networks with only local coupling (diffusive, to nearest neighbours) which are numerically found to support chimera states. One of the networks is analysed using a self-consistency argument in the continuum limit, and this is used to find the boundaries of existence of a chimera state in parameter space.

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Chimera states, in which a symmetric network of identical oscillators splits into two regions, one of coherent oscillators and one of incoherent, have been studied intensively over the past decade [1–3]. Spatial networks on which they have been studied include a one-dimensional ring [2, 4–8], a square domain without periodic boundary conditions [9–11], a torus [12, 13] and a sphere [14]. They have also been observed recently in a number of experimental settings [15–19]. Early investigations considered networks with nonlocal coupling (i.e. neither all-to-all with uniform strength, nor local coupling, via diffusion, for example) as chimeras were first reported in nonlocally coupled systems [1, 20]. Nonlocal coupling was at first thought to be essential for the existence of chimeras [7, 21], however, more recent results show that chimeras can occur in systems with purely global coupling [22–24].

Here we consider the opposite limit and address for the first time the question as to whether chimera states can exist in spatial networks with purely local coupling. We present three such networks in which this does occur. The idea behind the creation of the networks is straightforward and can be found in the early papers [11, 20]. Consider a general reaction-diffusion equation on a one-dimension spatial domain Ω with only local interactions via diffusion in one variable:

$$\frac{\partial u}{\partial t} = f(u) + v \tag{1}$$

$$\epsilon\frac{\partial v}{\partial t} = g(u) - v + \frac{\partial^2 v}{\partial x^2} \tag{2}$$

When ϵ is small and positive there is a separation of timescales in (1)-(2): u is "slow" and v is "fast." Taking the limit of infinitely fast dynamics for v, i.e. setting $\epsilon = 0$ in (2) one has

$$\left(1 - \frac{\partial^2}{\partial x^2}\right) v = g(u) \tag{3}$$

and if $h(x)$ is the Green's function associated with $\left(1 - \frac{\partial^2}{\partial x^2}\right)$ on Ω then we can solve (3) for v as

$$v(x) = \int_\Omega h(x-y) g(u(y))\, dy \tag{4}$$

and substituting this into (1) we obtain a closed nonlocal equation for u. [For $\Omega = \mathbb{R}$ and $\lim_{|x|\to\infty} h(x) = 0$, $h(x) = e^{-|x|}/2$, which does not have compact support.] Note that when $\epsilon \neq 0$ the only spatial interactions in (1)-(2) are local (via diffusion) and there are two dynamic variables (u and v). We will implement the network analogue of (1)-(2) but with ϵ small and nonzero in the expectation that the behaviour of interest when $\epsilon = 0$ persists for $\epsilon > 0$.

The first model we consider consists of N oscillators, equally-spaced on a domain of length L, with periodic boundary conditions. The state of oscillator j is described by two variables: $\theta_j \in [0, 2\pi)$ and $z_j \in \mathbb{C}$. (A complex variable is used to simplify presentation; we could equally well use two real variables.) The governing equations are

$$\frac{d\theta_j}{dt} = \omega_j - \mathrm{Re}\left(z_j e^{-i\theta_j}\right) \tag{5}$$

$$\epsilon\frac{dz_j}{dt} = Ae^{i(\theta_j+\beta)} - z_j + \frac{z_{j+1} - 2z_j + z_{j-1}}{(\Delta x)^2} \tag{6}$$

for $j = 1, 2 \ldots N$, where $\Delta x = L/N$ and A, β and ϵ are all constants. The ω_j are randomly chosen from a Lorentzian distribution with half-width-at-half-maximum σ centred at ω_0, namely

$$g(\omega) = \frac{\sigma/\pi}{(\omega - \omega_0)^2 + \sigma^2} \tag{7}$$

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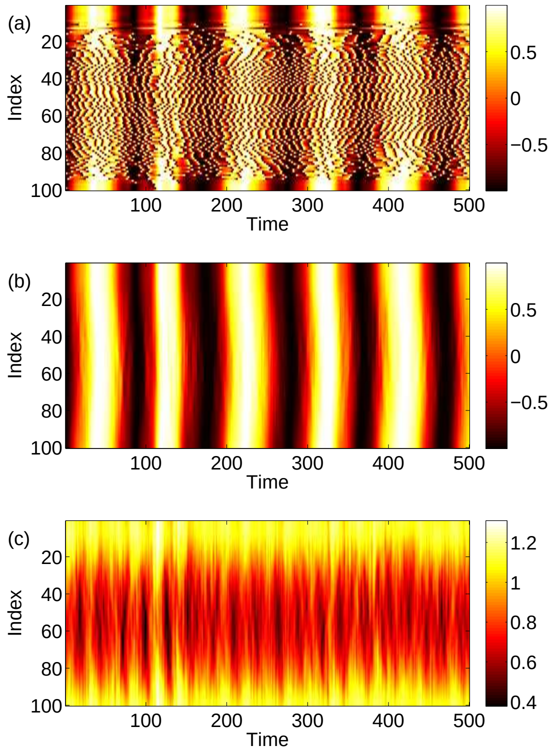

Figure 1: (Color online) Chimera solution of the system (5)-(6). (a): $\sin\theta_j$; (b): $\sin(\arg(z_j))$; (c): $|z_j|$. Parameters: $\omega_0 = 1, \sigma = 0.01, \epsilon = 0.2, A = 1.5, L = 2\pi, N = 100, \beta = 0.1$.

An example of the system's dynamics are shown in Fig. 1. The domain clearly splits into two regions, one showing coherent behaviour of the phases and the other, incoherent. This behaviour has been replicated in networks of up to $N = 1000$, so is not a small-N effect.

To understand the relationship between (5)-(6) and previously studied models we set $\epsilon = 0$ in (6). If z_j is the jth entry of the vector $\mathbf{z} \in \mathbb{C}^N$ and similarly for θ_j we can write (6) as

$$(I - D)\mathbf{z} = Ae^{i(\boldsymbol{\theta}+\beta)} \tag{8}$$

where I is the $N \times N$ identity matrix and D is the matrix representation of the classical second difference operator on N points with periodic boundary conditions. Defining $G = (I - D)^{-1}$ we have

$$z_j = A\sum_{k=1}^{N} G_{jk} e^{i(\theta_k+\beta)} \tag{9}$$

where G_{jk} is the jkth element of G, and substituting (9) into (5) we obtain

$$\frac{d\theta_j}{dt} = \omega_j - A\sum_{k=1}^{N} G_{jk}\cos(\theta_j - \theta_k - \beta) \tag{10}$$

Since $I - D$ is circulant, so is G, i.e. G_{jk} is a function of only $|j-k|$[a], and (10) is thus of the same form as studied by a number of others [1, 2, 4–8]. An important property of (10) is that it is invariant with respect to a uniform shift of all phases: $\theta_j \mapsto \theta_j + \gamma$ for all j where γ is some constant. This implies that (10) can be studied, without loss of generality, in a rotating coordinate frame where $\omega_0 = 0$, i.e. the actual value of ω_0 in (10) is irrelevant. This is not the case for (5)-(6) when $\epsilon \neq 0$ (although (5)-(6) is invariant under the simultaneous shift: $\theta_j \mapsto \theta_j + \gamma$ and $z_j \mapsto z_j e^{i\gamma}$ for all j). It is also clear that (10) is an N-dimensional dynamical system, while (5)-(6) is $3N$-dimensional, once real and imaginary parts of $\mathbf{z}$ are taken.

To analyse the chimera seen in (5)-(6) we use a self-consistency argument similar to that in [1, 2, 5]. We first move to a rotating coordinate frame, letting $\phi_j \equiv \theta_j - \Omega t$ and $y_j \equiv z_j e^{-i\Omega t}$, where Ω is to be determined, and then take the limit $N \to \infty$, to obtain

$$\frac{\partial \phi}{\partial t} = \omega - \mathrm{Re}\left(ye^{-i\phi}\right) - \Omega \tag{11}$$

$$\epsilon\frac{\partial y}{\partial t} = Ae^{i(\phi+\beta)} - y + \frac{\partial^2 y}{\partial x^2} - i\epsilon\Omega y \tag{12}$$

We now search for solutions of (11)-(12) for which y is stationary, i.e. just a function of space. We let such a solution be $y = R(x)e^{i\Theta(x)}$. Since y is constant we can use (11) to determine the dynamics of ϕ for any y and ω: if $|R| > |\omega - \Omega|$ then ϕ will tend to a stable fixed point of (11) whereas if $|R| < |\omega - \Omega|$ then ϕ will drift monotonically. To obtain a stationary solution of (12) we replace $e^{i\phi}$ by its expected value, calculated using the density of ϕ, which is inversely proportional to its velocity (given by (11)). So (keeping mind that ω is random variable) we need to solve

$$0 = Ae^{i\beta}\int_{-\infty}^{\infty}\int_{0}^{2\pi} e^{i\phi}p(\phi|\omega)g(\omega)d\phi\, d\omega - y + \frac{\partial^2 y}{\partial x^2} - i\epsilon\Omega y \tag{13}$$

where the density of ϕ given ω is

$$p(\phi|\omega) = \frac{\sqrt{(\omega-\Omega)^2 - R^2}}{2\pi|\omega - \Omega - R\cos(\Theta - \phi)|} \tag{14}$$

and $g(\omega)$ is given by (7). Evaluating the integrals in (13) we obtain

$$\frac{Ae^{i(\Theta+\beta)}}{R}\left[\omega_0 + i\sigma - \Omega - \sqrt{(\omega_0 + i\sigma - \Omega)^2 - R^2}\right] - \left(1 + i\epsilon\Omega - \frac{\partial^2}{\partial x^2}\right)Re^{i\Theta} = 0 \tag{15}$$

We determine R, Θ and Ω by simultaneously solving (15) and the scalar equation $\Theta(0) = 0$; this equation amounts to choosing the origin of the rotating coordinate frame.

[a] Explicitly, $G_{jk} = N^{-1}\sum_{r=0}^{N-1}\exp(-2\pi i r|j-k|/N)/\{1 + 2[1 - \cos(2\pi r/N)]/(\Delta x)^2\}$ [25], which is non-zero for all j, k.

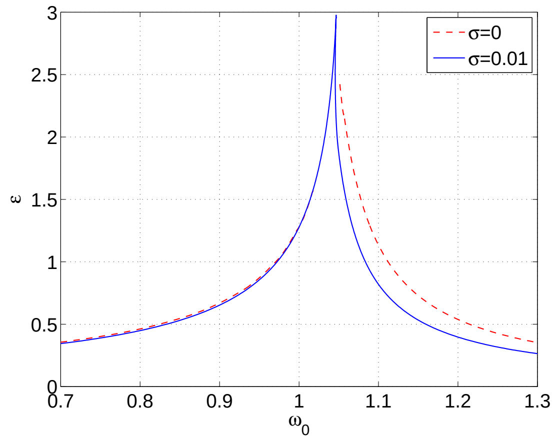

Figure 2: Saddle-node bifurcation of solutions of (15). Chimera solutions of (5)-(6) are stable below the curves. Parameters: $A = 1.5, L = 2\pi, \beta = 0.1, \sigma = 0.01$ (blue solid) and $\sigma = 0$ (red dashed).

Following solutions of (15) as ω_0 and ϵ are varied we find that two solutions are destroyed in a saddle-node bifurcation on the solid blue curve shown in Fig. 2 when $\sigma = 0.01$. Although our self-consistency argument gives no information about the stability of solutions (unlike the continuum theory in [4, 6, 8]) quasistatic sweeps through parameter space indicate that the solid curve in Fig. 2 does indeed mark the boundary of stable chimeras in the system (5)-(6). If ϵ is increased past the boundary in Fig. 2 when ω_0 is to the left of the cusp (at $\omega_0 \approx 1.04$) the system (5)-(6) moves to the almost synchronous state, whereas to the right of the cusp the system moves to a spatially-disordered state, and the almost synchronous state seems unstable here. The results above are for a hetereogeneous network ($\sigma = 0.01$) but following the saddle-node bifurcation for $\sigma = 0$ (i.e. identical oscillators) we obtain qualitatively the same result, as shown by the red dashed curve in Fig. 2. (Numerical difficulties prevented continuation of this curve through the cusp.)

Note that setting $\epsilon = 0$ in (15) one finds that Ω only appears in the combination $\omega_0 - \Omega$, i.e. only the *relative* frequency $\omega_0 - \Omega$ is unknown, and ω_0 can be set to any value (e.g. zero). This is not the case when $\epsilon \neq 0$, as is clearly seen in the dependence of existence of chimeras on the value of ω_0 in Fig. 2.

As a second example we consider a network of Stuart-Landau oscillators, each of which can be thought of as the normal form of a supercritical Hopf bifurcation, with purely local coupling through a second complex variable. Using Stuart-Landau oscillators as opposed to the phase oscillators above introduces an amplitude variable to the oscillator dynamics. As above we have N oscillators equally-spaced on a domain of length 1 with periodic boundary conditions. The equations are

$$\frac{dA_j}{dt} = (1 + i\omega_0)A_j - (1 + ib)|A_j|^2 A_j + K(1 + ia)(Z_j - A_j) \tag{16}$$

$$\epsilon\frac{dZ_j}{dt} = A_j - Z_j + \frac{Z_{j+1} - 2Z_j + Z_{j-1}}{16(\Delta x)^2} \tag{17}$$

for $j = 1, 2 \ldots N$ where $A_j, Z_j \in \mathbb{C}$ and ω_0, a, b, K and ϵ are real parameters and $\Delta x = 1/N$. Note that the oscillators are identical. A chimera state for this system is shown in Fig. 3. To understand the connection with previously-studied models, setting $\epsilon = 0$ in (17) and then taking the limit $N \to \infty$ we obtain

$$\left(1 - \frac{1}{16}\frac{\partial^2}{\partial x^2}\right) Z(x,t) = A(x,t) \tag{18}$$

If $h(x)$ is the Green's function for $\left(1 - \frac{1}{16}\frac{\partial^2}{\partial x^2}\right)$ on a periodic domain of length 1, then (16) becomes

$$\frac{\partial A(x,t)}{\partial t} = (1 + i\omega_0)A(x,t) - (1 + ib)|A(x,t)|^2 A(x,t) + K(1 + ia)\left[\int_0^1 h(x-y)A(y,t)\, dy - A(x,t)\right] \tag{19}$$

which is the nonlocally coupled complex Ginzburg-Landau equation for just the variable A, as studied by [1]. Then assuming that K is small one finds a scale separation between the amplitude and phase dynamics of A and upon setting $|A| = 1$ the phase dynamics can be written in a nonlocally coupled form [1, 3].

As a third model we consider a heterogeneous network of oscillators, each described by an angular variable and a real variable. The angular variables have the form of Winfree oscillators [26–28]. The model is

$$\frac{d\theta_j}{dt} = \omega_j + \kappa Q(\theta_j)u_j \tag{20}$$

$$\epsilon\frac{du_j}{dt} = P_n(\theta_j) - u_j + \frac{u_{j+1} - 2u_j + u_{j-1}}{(\Delta x)^2} \tag{21}$$

for $j = 1, 2 \ldots N$ where $Q(\theta) = \sin\beta - \sin(\theta + \beta)$, κ, β and ϵ are parameters, $P_n(\theta) = a_n(1 + \cos\theta)^n$ where $n \geq 1$ is an integer and $a_n = 2^n(n!)^2/(2n)!$ (so that $\int_0^{2\pi} P_n(\theta)d\theta = 2\pi$) and $\Delta x = L/N$. The ω_j are randomly chosen from a normal distribution with mean ω_0 and standard deviation σ. $Q(\theta)$ is the phase response curve of the oscillator and can be measured experimentally for a neuron, for example [29].

A chimera state for (20)-(21) is shown in Fig. 4. Setting $\epsilon = 0$ in (21) and solving for the u_j one would obtain a nonlocally coupled network of Winfree oscillators. Chimeras have been found in a network of two populations of Winfree oscillators [26, 30] but a truly nonlocally coupled network has apparently not yet been studied. Although the results in Fig. 4 are for a heterogeneous

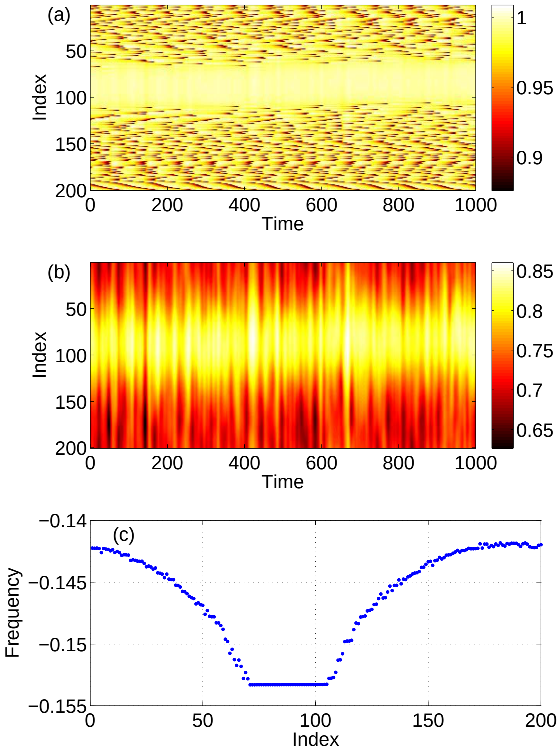

Figure 3: (Color online) Chimera solution of the system (16)-(17). (a): $|A_j|$; (b): $|Z_j|$; (c): average rotation frequency of the A_j over a simulation of duration 2000 time units. Parameters: $\omega_0 = 0, \epsilon = 0.01, a = -1, b = 1, K = 0.1, N = 200$.

network, similar stable chimera states are also observed for a network of identical oscillators (not shown).

We have presented three one-dimensional networks, where each node is described by one variable which has a phase associated with it and a second variable which is coupled in a diffusive fashion to just its two nearest neighbours. All networks have the same structure and show chimera states over some range of parameters. All have a small parameter (ϵ) which controls the time scale of the diffusing variable, so can be thought of as slow-fast systems [31]. As far as we are aware, this is the first demonstration of the existence of chimeras in networks with purely local coupling, as opposed to all previous studies which have used either nonlocal or global coupling [3].

We have not given any stability analysis of the models presented here, only a self-consistency argument for the first model. Chimeras in systems of the form (10) have been studied by passing to the continuum limit ($N \to \infty$) and analysing the resulting continuity equation using the Ott/Antonsen ansatz [4, 6, 8, 32, 33]. However, it does not seem that such an approach could be used to study the models presented here due to the dynamics of the extra variables.

Regarding experimental implementation, note that the nonlocal coupling in the experiments reported in [16, 18] was implemented by computer, i.e. the experiments were hybrid physical/computer. The models presented here — while being caricatures of physical systems — have only local, nearest-neighbour diffusive-like coupling. Since diffusion is ubiquitous in spatially extended systems of reacting chemicals, the most natural system in which to implement networks of the form discussed here (without a computer) may be in arrays of microsopic chemical oscillators [34–36].

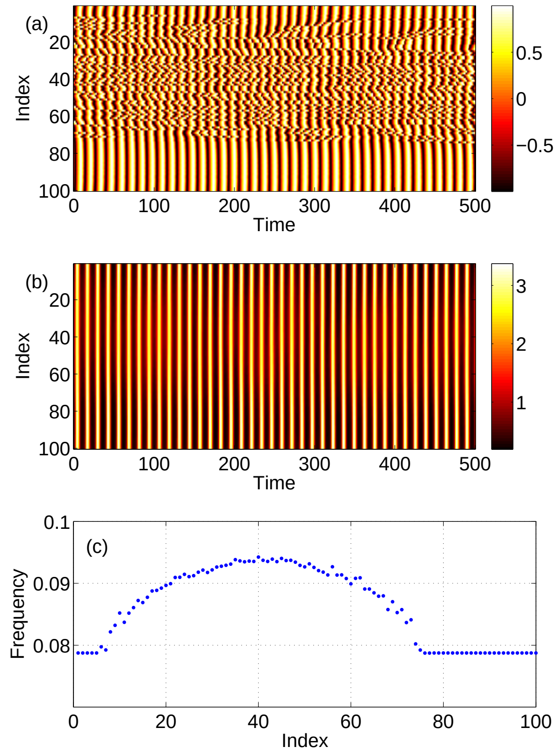

Figure 4: (Color online) Chimera solution of the system (20)-(21). (a): $\sin \theta_j$; (b): u_j; (c): average rotation frequency of the θ_j over a simulation of duration 2000 time units. Parameters: $\omega_0 = 0.3, \sigma = 0.001, n = 4, L = 4, \kappa = 0.4, \beta = \pi/2 - 0.2, \epsilon = 0.1, N = 100$.

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