Rotating waves in rings of coupled oscillators

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Abstract

In this paper we discuss the types of stable oscillation created via Hopf bifurcations for a ring of identical nonlinear oscillators, each of which is diffusively and symmetrically coupled to both its neighbours, and which, when uncoupled, undergo a supercritical Hopf bifurcation creating a stable periodic orbit as a parameter, λ, is increased.

We show that for small enough coupling, the only stable rotating waves produced are either one or a conjugate pair, depending on the parity of the number of oscillators in the ring and the sign of the coupling constant, and that the magnitude of the phase difference between neighbouring oscillators for these rotating waves is either zero (i.e. the oscillators are synchronised) or the maximum possible, depending on the sign of the coupling constant. These branches of rotating waves are produced supercritically.

1 Introduction

The behaviour of rings of identical oscillators has been studied by a number of authors [1, 2, 3, 4, 7, 11]. Such rings are used in the biological world to model such structures as the cross–section through a plant stem [9] or an intestine, the ring of tissue from which the petals of a flower emanate [11], or a ring of neuronal oscillators, each of which controls the motion of one leg of an animal during locomotion [3]. In the world of physics, they have been used to model rings of semiconductor lasers [10].

Various approaches have been taken to try and understand the types of possible behaviour in such rings. One approach, taken by Ashwin and Swift [2], is to assume that the coupling between oscillators is weak relative to the attractiveness of the periodic orbit that exists in the phase space of each oscillator. This assumption allows a decoupling of the dynamics for the phases of each of the oscillators from the dynamics of their amplitudes. Since each oscillator has only one phase variable associated with it, this reduces the dynamical system from one in R^{Nn} to one on T^N, where N is the number of oscillators in the ring, the phase space for each uncoupled oscillator is R^n, and T^N is the N–torus. The symmetry group of N oscillators coupled in a ring with no preferred direction around the ring is the dihedral group $\mathbf{D}_N$, i.e. the group of symmetries of a regular N–gon. The flow on T^N inherits this symmetry, so the problem of finding the possible dynamics for the ring of identical oscillators has reduced to that of finding possible types of $\mathbf{D}_N$–equivariant flows on T^N. (Under the assumption that the differences in phase between oscillators vary on a slower time scale than the oscillatory motion of each oscillator, the $\mathbf{D}_N$–equivariant flow on T^N can be *averaged*, resulting in a $\mathbf{D}_N$–equivariant flow on T^{N-1}.)

Grasman and Jansen [7] use a similar weak coupling assumption, but consider perturbing away from a ring of relaxation oscillators that have an infinite rate of relaxation.

The other approach that has had some success is assume that each oscillator is parametrised in such a way that it undergoes a Hopf bifurcation as a parameter is varied, and to use the theory of Hopf bifurcations, with or without symmetry, to determine the types of oscillation produced. This was the approach used by Collins and Stewart [3] in relation to the modelling of animal gaits. These authors used results from the theory of Hopf bifurcation with either $\mathbf{D}_N$ or $\mathbf{Z}_N$ symmetry [5, 6] to determine the possible types of oscillation that could exist, purely as a result of the symmetry of the network. As they mention, the question of the stability of any of these types of oscillation depends on the exact form of the oscillators and coupling used. In this paper we extend their results to show that for quite general oscillators, one specific branch of oscillations is stable.

When the linearisation of the coupled system about a fixed point has a simple pair of purely imaginary eigenvalues, the theory of simple Hopf bifurcation (see e.g. [8, 12]) can be used to determine the type of oscillation produced in that bifurcation, and by appropriate centre manifold reduction calculations, the direction of branching and the consequent stability of the orbit can be determined. This is the approach taken by Alexander and Auchmuty [1], among others. However, for systems with $\mathbf{D}_N$ symmetry, most of the eigenvalues are forced to appear as pairs, and simple Hopf bifurcation theory can no longer be used. This problem can sometimes be resolved in particular cases. For example, in Silber et al. [10], the equations governing the behaviour of a ring of semiconductor lasers are such that periodic solutions can be found explicitly, and these can be linearised about to give their stabilities. Ermentrout [4] studied the case of weak coupling near a Hopf bifurcation and found families of rotating waves and necessary conditions for their stabilities in the limit of large N. His results are a special case of the results we derive.

The tools we use are centre manifold and normal form theory. For a general coupled system (1) we determine which Hopf bifurcation may produce a stable periodic orbit, perform a centre manifold reduction at this bifurcation point, and then do the normal form calculations necessary to determine the sub– or supercriticality of the branches produced here and their stabilities. For more details on centre manifold and normal form theory, see [8, 12].

2 Presentation of system

We assume that our system of N coupled oscillators in a ring is governed by equations of the form

$$\dot{z}_j \equiv \frac{dz_j}{dt} = (\lambda + i\Omega)z_j + F_2(z_j, \bar{z}_j) + F_3(z_j, \bar{z}_j) + (\epsilon_r + i\epsilon_i)(2z_j - z_{j+1} - z_{j-1}) \tag{1}$$

where $z_j \in$ C; $\lambda, \Omega, \epsilon_r, \epsilon_i \in$ R, $i^2 = -1$, the subscripts (which label the oscillators) are taken mod N, and F_2 and F_3 contain the second and third order terms, respectively, in the Taylor series expansion of the vector field of an uncoupled oscillator. To be more explicit, we write

$$F_2(z_j, \bar{z}_j) = \alpha_1 z_j^2 + \alpha_2 z_j \bar{z}_j + \alpha_3 \bar{z}_j^2$$

and

$$F_3(z_j, \bar{z}_j) = \beta_1 z_j^3 + \beta_2 z_j^2 \bar{z}_j + \beta_3 z_j \bar{z}_j^2 + \beta_4 \bar{z}_j^3$$

where the α's and β's are complex constants. We take λ as our bifurcation parameter, which we assume to be close to zero. The α's and β's will typically depend on λ, but because we are only concerned with the case $|\lambda| \ll 1$, we fix them at the value they have when $\lambda = 0$.

We assume that each uncoupled oscillator undergoes a supercritical Hopf bifurcation as λ increases through zero. This means that the normal form coefficient, a, for each oscillator is negative. For a system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} \qquad x, y \in \mathrm{R}$$

with $f(0,0) = g(0,0) = 0$ and $Df(0,0) = Dg(0,0) = 0$ at a simple Hopf bifurcation, the expression for a is [8]

$$a = \frac{1}{16}[f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}] + \frac{1}{16\omega}[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}]$$

where subscripts indicate partial differentiation with respect to that variable. It is straight–forward to calculate this quantity in terms of the α's and β's in (1):

$$a = Re\{\beta_2\} - \frac{Im\{\alpha_1\}Re\{\alpha_2\} + Re\{\alpha_1\}Im\{\alpha_2\}}{\Omega} = Re\{\beta_2\} - \frac{Im\{\alpha_1\alpha_2\}}{\Omega} \tag{2}$$

Note that this expression is linear in coefficients of cubic terms in the Taylor series and quadratic in coefficients of second order terms — this fact will be used later.

The Jacobian of (1) evaluated at the origin is circulant, so we can easily find its eigenvalues and eigenvectors. We use the eigenvectors to diagonalise the linear part of the system, defining a new coordinate system w by $w = Az$, where

$$w = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_{N-1} \\ w_N \end{pmatrix} \quad A = \begin{pmatrix} 1 & \xi^{(N-1)(N-1)} & \dots & \xi^{2(N-1)} & \xi^{N-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \xi^{2(N-1)} & \dots & \xi^4 & \xi^2 \\ 1 & \xi^{N-1} & \dots & \xi^2 & \xi \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix} \quad z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{N-1} \\ z_N \end{pmatrix} \quad (3)$$

and $\xi = e^{2\pi i/N}$, i.e. $A_{jk} = \xi^{(N-j)(N-k+1)}$. After this transformation the linear part of the $\dot{w}$ equation is diagonal with entries of $\lambda + i\Omega + 2(\epsilon_r + i\epsilon_i)\left[1 - \cos\left(\frac{2\pi j}{N}\right)\right]$ for $j = 0, \dots, N-1$ with the $j = 0$ entry at bottom right and the $j = N-1$ entry at top left. From the form of these diagonal terms we see that there is a Hopf bifurcation (which may be simple or double) from the origin when $\lambda + 2\epsilon_r\left[1 - \cos\left(\frac{2\pi j}{N}\right)\right] = 0$ for some $0 \le j \le N-1$.

For $\lambda < -4|\epsilon_r|$ all of the eigenvalues of the Jacobian at the origin have negative real parts and the origin is stable, so we look for the first Hopf bifurcation to occur as λ is increased, which we hope will produce a stable branch of oscillations. Once this Hopf bifurcation has occurred the origin is unstable, and all of the orbits subsequently created from it in Hopf bifurcations will necessarily be unstable. First we take the case $\epsilon_r < 0$; the analysis of the case $\epsilon_r > 0$ depends on whether N is even or odd, and we deal with these cases in subsequent sections.

3 $\epsilon_r < 0$

When $\epsilon_r < 0$ the first Hopf bifurcation to occur as λ is increased is the simple one with $j = 0$ at $\lambda = 0$, irrespective of the value of N. At this bifurcation the centre manifold in the w coordinate system is in the w_N direction and since the Jacobian is diagonal and all of its other eigenvalues have negative real part, the dynamics for all the other components of w are dominated by exponential contraction onto the centre manifold. We know from (3) that

$$w_N = \sum_{j=1}^{N} z_j,$$

so using this and (1) we have

$$\begin{aligned}
\dot{w}_N &= \sum_{j=1}^{N} \dot{z}_j \\
&= \sum_{j=1}^{N} \{(\lambda + i\Omega) z_j + F_2(z_j, \bar{z}_j) + F_3(z_j, \bar{z}_j) + (\epsilon_r + i\epsilon_i)(2z_j - z_{j-1} - z_{j+1})\} \\
&= (\lambda + i\Omega) w_N + \sum_{j=1}^{N} \{F_2(z_j, \bar{z}_j) + F_3(z_j, \bar{z}_j)\} \qquad (4) \\
&= (\lambda + i\Omega) w_N + \frac{F_2(w_N, \bar{w}_N)}{N} + [\ldots 2 \ldots] + \frac{F_3(w_N, \bar{w}_N)}{N^2} + [\ldots 3 \ldots]
\end{aligned}$$

where $[\ldots 2 \ldots]$ represents second order terms in $w_1, \bar{w}_1, \ldots, w_{N-1}, \bar{w}_{N-1}$ and $[\ldots 3 \ldots]$ represents all cubic terms that include at least one of $w_1, \bar{w}_1, \ldots, w_{N-1}, \bar{w}_{N-1}$. The second and fourth terms in the last line of (4) were obtained by using the inverse of A,

$$A^{-1}_{jk} = \left(\frac{1}{N}\right) \xi^{(j-1)(N-k)},$$

so that

$$z_j = \sum_{k=1}^{N} A^{-1}_{jk} w_k = \frac{1}{N} \sum_{k=1}^{N} \xi^{(j-1)(N-k)} w_k \qquad (5)$$

and substituting into the expressions for $F_2(z_j, \bar{z}_j)$ and $F_3(z_j, \bar{z}_j)$.

There is a subtle point here regarding the difference between $[\ldots 2 \ldots]$ and $[\ldots 3 \ldots]$. We might expect there to be second order terms in (4) of the form $w_N w_k$, $w_N \bar{w}_k$, $\bar{w}_N w_k$ or $\bar{w}_N \bar{w}_k$ for some $k \neq N$, and when we then perform a centre manifold reduction (in which we write w_k and $\bar{w}_k$ as a sum of second–order terms in w_N and $\bar{w}_N$) these terms will be third order in $w_N, \bar{w}_N$. However, it is possible to calculate the coefficients of the second order terms in $w_N w_k$, $w_N \bar{w}_k$, $\bar{w}_N w_k$ and $\bar{w}_N \bar{w}_k$ for any $k \neq N$, and they are all zero. Hence, when we perform the centre manifold reduction all terms in $[\ldots 2 \ldots]$ are fourth order in $w_N, \bar{w}_N$ and will therefore be ignored. For a similar reason, all terms in [...3...] will be of order *at least* 4 (possibly up to 6) and will similarly be ignored. Thus, after performing the centre manifold reduction we can write (4) as

$$\dot{w}_N = (\lambda + i\Omega)w_N + \frac{F_2(w_N, \bar{w}_N)}{N} + \frac{F_3(w_N, \bar{w}_N)}{N^2} + O(|w_N|^4) \tag{6}$$

Comparing this with (1) at $(\epsilon_r, \epsilon_i) = (0, 0)$ we see that to third order they are identical except that the coefficients of the second order terms in (6) have been divided by N and the third order ones by N^2. Going back to the expression for a (equation (2)), we see that the value of a for (6) is equal to that for an uncoupled oscillator divided by N^2. Since only the sign of a is important, we see that in the dynamics restricted to the centre manifold, (6), there is a supercritical Hopf bifurcation as λ increases through 0 when $\epsilon_r < 0$. To see how this manifests itself in the oscillators, we use the inverse of A, (5). Since w_N is small at the onset of oscillation and the w_k for $k \neq N$ are second order in w_N, we can ignore their contribution to the z_j. From (5) we see that $z_j \sim w_N$ for all j, i.e. this branch of orbits manifests itself as the *completely synchronised* state, $z_j = z_k$ for all j, k. The next case we do is $\epsilon_r > 0$, N even.

4 $\epsilon_r > 0$, N even

In this case the first Hopf bifurcation to occur as λ is increased is the simple one for $j = N/2$ at $\lambda = -4\epsilon_r$. The centre manifold is in the $w_{N/2}$ direction and as above, the dynamics in the other directions of w are dominated by exponential contraction onto the centre manifold. From (3) we have

$$w_{N/2} = \sum_{j=1}^{N} \left(A_{\frac{N}{2}j}\right) z_j = \sum_{j=1}^{N} (-1)^{N-j+1} z_j$$

so using (1) we obtain

$$
\begin{aligned}
\dot{w}_{N/2} &= \sum_{j=1}^{N}(-1)^{N-j+1}\dot{z}_j \\
&= \sum_{j=1}^{N}(-1)^{N-j+1}\{(\lambda+i\Omega)z_j + F_2(z_j,\bar{z}_j) + F_3(z_j,\bar{z}_j) + (\epsilon_r+i\epsilon_i)(2z_j - z_{j+1} - z_{j-1})\} \\
&= [\lambda+i\Omega+4(\epsilon_r+i\epsilon_i)]w_{N/2} + \sum_{j=1}^{N}(-1)^{N-j+1}\{F_2(z_j,\bar{z}_j)+F_3(z_j,\bar{z}_j)\} \\
&= [\lambda+i\Omega+4(\epsilon_r+i\epsilon_i)]w_{N/2} \\
&+ \frac{1}{N}\left[2\alpha_1 w_{N/2}w_N + \alpha_2(w_{N/2}\bar{w}_N + \bar{w}_{N/2}w_N) + 2\alpha_3\bar{w}_{N/2}\bar{w}_N\right] \\
&+ [\ldots 2\ldots] + \frac{F_3(w_{N/2},\bar{w}_{N/2})}{N^2} + [\ldots 3\ldots]
\end{aligned}
\tag{7}
$$

where $[\ldots 2\ldots]$ now represents second order terms in $w_1, \bar{w}_1, \ldots, w_N, \bar{w}_N$ that have *no* factors of $w_{N/2}$ or $\bar{w}_{N/2}$, and $[\ldots 3\ldots]$ represents cubic terms not composed exclusively of $w_{N/2}$ and $\bar{w}_{N/2}$. After performing the centre manifold reduction, terms in $[\ldots 2\ldots]$ and $[\ldots 3\ldots]$ will be of order at least 4 in $w_{N/2}$ and $\bar{w}_{N/2}$, and will therefore be ignored from now on. The last step is to actually perform (part of) the centre manifold reduction in order to write w_N (and hence $\bar{w}_N$) as a function of $w_{N/2}$ and $\bar{w}_{N/2}$ so that we can substitute these expressions into (7) and obtain an equation for motion on the centre manifold.

We start by writing

$$
w_N = g(w_{N/2},\bar{w}_{N/2}) = \sigma_1 w_{N/2}^2 + \sigma_2 w_{N/2}\bar{w}_{N/2} + \sigma_3\bar{w}_{N/2}^2 \tag{8}
$$

as an approximation to the centre manifold, where $\sigma_{1,2,3} \in \mathrm{C}$ are (as yet) unknown. We find them by equating two equivalent expressions for $\dot{w}_N$:

$$
\dot{w}_N = \frac{\partial g}{\partial w_{N/2}}\dot{w}_{N/2} + \frac{\partial g}{\partial \bar{w}_{N/2}}\dot{\bar{w}}_{N/2} = (\lambda+i\Omega)g(w_{N/2},\bar{w}_{N/2}) + f_2(w_{N/2},\bar{w}_{N/2}) \tag{9}
$$

where $f_2(w_{N/2},\bar{w}_{N/2})$ are the terms in the $[\ldots 2\ldots]$ of (4) involving only $w_{N/2}$ and $\bar{w}_{N/2}$. It is straight–forward to show that $f_2(w_{N/2},\bar{w}_{N/2})$ is in fact $F_2(w_{N/2},\bar{w}_{N/2})/N$.

Thus (9) becomes

$$\begin{aligned} & [2\sigma_1 w_{N/2} + \sigma_2 \bar{w}_{N/2}] \times [\lambda + i\Omega + 4(\epsilon_r + i\epsilon_i)] w_{N/2} \\ + & [\sigma_2 w_{N/2} + 2\sigma_3 \bar{w}_{N/2}] \times [\lambda - i\Omega + 4(\epsilon_r - i\epsilon_i)] \bar{w}_{N/2} \\ = & (\lambda + i\Omega) \times [\sigma_1 w_{N/2}^2 + \sigma_2 w_{N/2} \bar{w}_{N/2} + \sigma_3 \bar{w}_{N/2}^2] + F_2(w_{N/2}, \bar{w}_{N/2})/N \end{aligned}$$

We equate coefficients of second order terms in $w_{N/2}$ and $\bar{w}_{N/2}$ in this expression to get

$$\sigma_1 = \frac{\alpha_1}{N[\lambda + i\Omega + 8(\epsilon_r + i\epsilon_i)]}, \quad \sigma_2 = \frac{\alpha_2}{N[\lambda - i\Omega + 8\epsilon_r]}, \quad \sigma_3 = \frac{\alpha_3}{N[\lambda - 3i\Omega + 8(\epsilon_r - i\epsilon_i)]}$$

Now that we have found σ_1, σ_2 and σ_3, we can substitute the expression for w_N, (8), into (7) to obtain the equation for motion on the centre manifold. It is

$$\dot{w}_{N/2} = [\lambda + i\Omega + 4(\epsilon_r + i\epsilon_i)] w_{N/2} + C w_{N/2}^2 \bar{w}_{N/2} + [\ldots iii \ldots]$$

where

$$C = \frac{\beta_2}{N^2} + \frac{2\alpha_1\sigma_2}{N} + \frac{\alpha_2}{N}(\bar{\sigma}_2 + \sigma_1) + \frac{2\alpha_3\bar{\sigma}_3}{N}$$

and $[\ldots iii \ldots]$ represents all terms of order 3 or more in the Taylor series excluding the term in $w_{N/2}^2 \bar{w}_{N/2}$. If the real part of C is negative at the bifurcation value ($\lambda = -4\epsilon_r$), we will have a supercritical Hopf bifurcation in the $w_{N/2}$ direction as λ increases through $-4\epsilon_r$.

For small ϵ_r, ϵ_i we can expand C at $\lambda = -4\epsilon_r$ in powers of ϵ_r and ϵ_i as

$$\begin{aligned} C &= \frac{1}{N^2}\left[\beta_2 + i\frac{\alpha_1\alpha_2}{\Omega} - i\frac{\alpha_2\bar{\alpha}_2}{\Omega} - i\frac{2\alpha_3\bar{\alpha}_3}{3\Omega}\right] \\ &+ \frac{\epsilon_r}{N^2\Omega^2}\left[12\alpha_1\alpha_2 + 4\alpha_2\bar{\alpha}_2 + \frac{8\alpha_3\bar{\alpha}_3}{9}\right] + \frac{\epsilon_i}{N^2\Omega^2}\left[8i\alpha_1\alpha_2 + \frac{16i\alpha_3\bar{\alpha}_3}{9}\right] + O(|\epsilon_r, \epsilon_i|^2) \end{aligned}$$

and thus

$$\begin{aligned} Re\{C\} &= \frac{1}{N^2}\left[a + \frac{\epsilon_r}{\Omega^2}\left(12 Re\{\alpha_1\alpha_2\} + 4\alpha_2\bar{\alpha}_2 + \frac{8\alpha_3\bar{\alpha}_3}{9}\right) - \frac{\epsilon_i}{\Omega^2}\left(8 Im\{\alpha_1\alpha_2\}\right)\right] \\ &+ O(|\epsilon_r, \epsilon_i|^2) \end{aligned}$$

where a is as given in (2). Thus for small enough ϵ_r and ϵ_i, $Re\{C\}$ will be negative, and we will have a supercritical Hopf bifurcation in the $w_{N/2}$ direction. As before, we use (5) to see how this branch of oscillations appears in the oscillators. We find that $z_j \sim (-1)^{j-1} w_{N/2}$, i.e. $z_{j+1} = -z_j$ for all j and thus neighbouring oscillators are half a period out of phase with one another. We call this the *exact antiphase* state. The last case to do is $\epsilon_r > 0$, N odd.

5 $\epsilon_r > 0$, N odd

In this case the first Hopf bifurcation to occur as λ increases is double, corresponding to both $j = (N+1)/2$ and $j = (N-1)/2$, at $\lambda = -2\epsilon_r[1 + \cos(\frac{\pi}{N})]$. For notational convenience, we define $p \equiv (N-1)/2$ and $q \equiv (N+1)/2$. At this bifurcation the centre manifold is four–dimensional, in the directions of w_p and w_q, and there are generically three branches of periodic orbits (labelled $\tilde{\mathbf{Z}}_N, \mathbf{Z}_2(\kappa)$ and $\mathbf{Z}_2(\kappa, \pi)$ by Golubitsky et al. [6]) that emanate from the double Hopf bifurcation. The sub– or supercriticality of these branches and their stability or otherwise depends on the coefficients of the cubic terms in the normal form of the equations on the centre manifold. We derive these coefficients below for (1) in terms of the equation for an uncoupled oscillator.

Once we have these coefficients, we can compare them with those in the normal form of the $\mathbf{D}_N$ symmetric double Hopf bifurcation (see §3, Ch. XVIII of [6]) which we choose to write as follows:

$$
\begin{aligned}
\dot{u}_1 &= \mu u_1 + B|u_1|^2 u_1 + C|u_2|^2 u_1 + O(|u_1, u_2|^5) \\
\dot{u}_2 &= \mu u_2 + B|u_2|^2 u_2 + C|u_1|^2 u_2 + O(|u_1, u_2|^5)
\end{aligned}
\quad (10)
$$

where all parameters and variables (except time) are complex and $Re\{\mu\}$ is the bifurcation parameter. The difference between this presentation of the normal form and that in [6] is that we have collected all terms of order 5 or more in the "$O(|u_1, u_2|^5)$" term. We have done this because, although higher order terms are necessary to determine the stability of the $\mathbf{Z}_2(\kappa)$ and $\mathbf{Z}_2(\kappa, \pi)$ branches, we show below that for $|\epsilon_r|$ and $|\epsilon_i|$ small enough there is a bifurcation to a stable $\tilde{\mathbf{Z}}_N$ branch, the direction of bifurcation and stability of this branch being completely determined by the coefficients of the cubic terms, B and C; thus the exact form of the higher order terms is not relevant.

The normal form has three types of nontrivial solution:

1. $u_1 = u_2$, which we associate with the $\mathbf{Z}_2(\kappa)$ orbit,

2. $u_1 = -u_2$, which we associate with the $\mathbf{Z}_2(\kappa, \pi)$ orbit, and

3. either $(u_1, u_2) = (u_1, 0)$ or $(u_1, u_2) = (0, u_2)$, both of which we associate with $\widetilde{\mathbf{Z}}_N$ orbits.

The bifurcation set for the normal form (10) is shown in Figure 1, which is reparametrised version of Figure 3.1 in Ch. XVIII of [6] showing how the bifurcation diagrams for (10) depend on the real parts of B and C. See [6] for more details.

Defining $\xi_- \equiv \xi^{(N-1)/2}$ and $\xi_+ \equiv \xi^{(N+1)/2}$, where $\xi = e^{2\pi i/N}$, we have, using (3)

$$w_q = \sum_{k=1}^{N} \xi_-^{N-k+1} z_k \quad \text{and} \quad w_p = \sum_{k=1}^{N} \xi_+^{N-k+1} z_k$$

Differentiating the first of these with respect to time and using (1) we have

$$\begin{aligned}
\dot{w}_q &= \sum_{k=1}^{N} \xi_-^{N-k+1} \{(\lambda + i\Omega) z_j + F_2(z_j, \bar{z}_j) + F_3(z_j, \bar{z}_j) + (\epsilon_r + i\epsilon_i)(2z_j - z_{j+1} - z_{j+1})\} \\
&= \{\lambda + i\Omega + 2(\epsilon_r + i\epsilon_i)[1 + \cos(\pi/N)]\} w_q + \sum_{k=1}^{N} \xi_-^{N-k+1} \{F_2(z_k, \bar{z}_k) + F_3(z_k, \bar{z}_k)\} \\
&= \{\lambda + i\Omega + 2(\epsilon_r + i\epsilon_i)[1 + \cos(\pi/N)]\} w_q \qquad (11) \\
&+ [2\alpha_1 w_q w_N + \alpha_2(w_q \bar{w}_N + \bar{w}_q w_1) + 2\alpha_3 \bar{w}_q \bar{w}_{N-1}]/N \\
&+ [2\alpha_1 w_p w_1 + \alpha_2(w_p \bar{w}_{N-1} + \bar{w}_p w_N) + 2\alpha_3 \bar{w}_p \bar{w}_N]/N + [\ldots ii \ldots] \\
&+ \frac{\beta_2}{N^2}[w_q^2 \bar{w}_q + 2 w_q w_p \bar{w}_p] + [\ldots iii \ldots]
\end{aligned}$$

where $[\ldots ii \ldots]$ represents second order terms with no factors of $w_q, \bar{w}_q, w_p$ or $\bar{w}_p$, and $[\ldots iii \ldots]$ represents cubic terms excluding those of the form $w_q^2 \bar{w}_q$ and $w_q w_p \bar{w}_p$. When the centre manifold reduction is performed terms in $[\ldots ii \ldots]$ and $[\ldots iii \ldots]$ will be of order at least 4 in $|w_p|$ and $|w_q|$, or if not, can be removed with normal form transformations, and will thus be ignored from now on. We obtain an expression analogous to (11) for $\dot{w}_p$, with w_p and w_q exchanged, as expected from the symmetry of the problem.

The next step is to perform the centre manifold reduction in order to get expressions for w_1, w_N and w_{N-1} in terms of $w_q, \bar{w}_q, w_p$ and $\bar{w}_p$ so that we can substitute them into (11). We write

$$\begin{aligned} w_1 = f(w_q, \bar{w}_q, w_p, \bar{w}_p) &= \gamma_1 w_p^2 + \gamma_2 w_p \bar{w}_p + \gamma_3 w_p w_q + \gamma_4 w_p \bar{w}_q + \gamma_5 \bar{w}_p^2 \\ &+ \gamma_6 \bar{w}_p w_q + \gamma_7 \bar{w}_p \bar{w}_q + \gamma_8 w_q^2 + \gamma_9 w_q \bar{w}_q + \gamma_{10} \bar{w}_q^2 \\ w_N = g(w_q, \bar{w}_q, w_p, \bar{w}_p) &= \theta_1 w_p^2 + \theta_2 w_p \bar{w}_p + \theta_3 w_p w_q + \theta_4 w_p \bar{w}_q + \theta_5 \bar{w}_p^2 \\ &+ \theta_6 \bar{w}_p w_q + \theta_7 \bar{w}_p \bar{w}_q + \theta_8 w_q^2 + \theta_9 w_q \bar{w}_q + \theta_{10} \bar{w}_q^2 \\ w_{N-1} = h(w_q, \bar{w}_q, w_p, \bar{w}_p) &= \nu_1 w_p^2 + \nu_2 w_p \bar{w}_p + \nu_3 w_p w_q + \nu_4 w_p \bar{w}_q + \nu_5 \bar{w}_p^2 \\ &+ \nu_6 \bar{w}_p w_q + \nu_7 \bar{w}_p \bar{w}_q + \nu_8 w_q^2 + \nu_9 w_q \bar{w}_q + \nu_{10} \bar{w}_q^2 \end{aligned} \tag{12}$$

where $\gamma_1, \dots, \nu_{10} \in \mathrm{C}$ are unknown coefficients. We find them in the usual way — writing equivalent expressions for each of $\dot{w}_1$, $\dot{w}_N$ and $\dot{w}_{N-1}$ and then equating coefficients of like powers of $w_q, \bar{w}_q, w_p$ and $\bar{w}_p$. By substituting the expressions (12) into (11), we can see that after the centre manifold reduction has been performed the coefficient of the term in $w_q^2 \bar{w}_q$ in (11) will be

$$B \equiv \frac{\beta_2}{N^2} + \frac{2\alpha_1 \theta_9 + \alpha_2(\bar{\theta}_9 + \gamma_8) + 2\alpha_3 \bar{\nu}_{10}}{N} \tag{13}$$

while that of the term in $w_q w_p \bar{w}_p$ will be

$$C \equiv \frac{2\beta_2}{N^2} + \frac{2\alpha_1 \theta_2 + \alpha_2 \bar{\theta}_2 + 2\alpha_1 \gamma_6 + \alpha_2(\bar{\nu}_4 + \theta_3) + 2\alpha_3 \bar{\theta}_7}{N} \tag{14}$$

(We have made the correspondence $u_1 = w_q$ and $u_2 = w_p$ for comparison between (11) and (10).) Actually doing the centre manifold reduction, i.e. finding $\gamma_1, \dots, \nu_{10}$, we see that at the double Hopf bifurcation (i.e. when $\lambda = -2\epsilon_r[1+\cos(\pi/N)]$)

$$
\begin{aligned}
\theta_9 &= \frac{i\alpha_2}{\Omega N} + \epsilon_r\left[\frac{2\alpha_2}{\Omega^2 N}(1+\cos(\pi/N))\right] + O(|\epsilon_r,\epsilon_i|^2)\\
\gamma_8 &= -\frac{i\alpha_1}{\Omega N} + \epsilon_r\left[\frac{2\alpha_1}{\Omega^2 N}(\cos(\pi/N)+\cos(2\pi/N))\right]\\
&+ \epsilon_i\left[\frac{2i\alpha_1}{N\Omega^2}(1+2\cos(\pi/N)+\cos(2\pi/N))\right] + O(|\epsilon_r,\epsilon_i|^2)\\
\nu_{10} &= \frac{i\alpha_3}{3\Omega N} + \epsilon_r\left[\frac{2\alpha_3}{9\Omega^2 N}(\cos(\pi/N)+\cos(2\pi/N))\right]\\
&+ \epsilon_i\left[\frac{2i\alpha_3}{9N\Omega^2}(\cos(2\pi/N)-2\cos(\pi/N)-3)\right] + O(|\epsilon_r,\epsilon_i|^2)\\
\theta_2 &= \frac{i\alpha_2}{\Omega N} + \epsilon_r\left[\frac{2\alpha_2}{\Omega^2 N}(1+\cos(\pi/N))\right] + O(|\epsilon_r,\epsilon_i|^2)\\
\gamma_6 &= \frac{i\alpha_2}{\Omega N} + \epsilon_r\left[\frac{2\alpha_2}{\Omega^2 N}(\cos(\pi/N)+\cos(2\pi/N))\right]\\
&+ \epsilon_i\left[\frac{2i\alpha_2}{\Omega^2 N}(\cos(2\pi/N)-1)\right] + O(|\epsilon_r,\epsilon_i|^2)\\
\nu_4 &= \frac{i\alpha_2}{\Omega N} + \epsilon_r\left[\frac{2\alpha_2}{\Omega^2 N}(\cos(\pi/N)+\cos(2\pi/N))\right]\\
&+ \epsilon_i\left[\frac{2i\alpha_2}{N\Omega^2}(\cos(2\pi/N)-1)\right] + O(|\epsilon_r,\epsilon_i|^2)\\
\theta_3 &= -\frac{2i\alpha_1}{\Omega N} + \epsilon_r\left[\frac{4\alpha_1}{\Omega^2 N}(1+\cos(\pi/N))\right] + \epsilon_i\left[\frac{8i\alpha_1}{\Omega^2 N}(1+\cos(\pi/N))\right] + O(|\epsilon_r,\epsilon_i|^2)\\
\theta_7 &= \frac{2i\alpha_3}{3\Omega N} + \epsilon_r\left[\frac{4\alpha_3}{9\Omega^2 N}(1+\cos(\pi/N))\right] + \epsilon_i\left[\frac{-8i\alpha_3}{9N\Omega^2}(1+\cos(\pi/N))\right] + O(|\epsilon_r,\epsilon_i|^2)
\end{aligned}
$$

for small $|\epsilon_r|$, $|\epsilon_i|$. Substituting these expansions into the expressions for B and C (13–14) we obtain

$$
\begin{aligned}
Re\{B\} &= \frac{1}{N^2}\left[Re\{\beta_2\} - \frac{Im\{\alpha_1\alpha_2\}}{\Omega}\right] \qquad (15)\\
&+ \frac{\epsilon_r}{N^2\Omega^2}\left[2Re\{\alpha_1\alpha_2\}\left(2+3\cos\left(\frac{\pi}{N}\right)+\cos\left(\frac{2\pi}{N}\right)\right)\right.\\
&+ \left. 2|\alpha_2|^2\left(1+\cos\left(\frac{\pi}{N}\right)\right) + \frac{4|\alpha_3|^2}{9}\left(\cos\left(\frac{\pi}{N}\right)+\cos\left(\frac{2\pi}{N}\right)\right)\right]\\
&- \frac{\epsilon_i}{N^2\Omega^2}\left[2Im\{\alpha_1\alpha_2\}\left(1+2\cos\left(\frac{\pi}{N}\right)+\cos\left(\frac{2\pi}{N}\right)\right)\right] + O(|\epsilon_r,\epsilon_i|^2)
\end{aligned}
$$

and

$$
\begin{aligned}
Re\{C\} &= \frac{2}{N^2}\left[Re\{\beta_2\} - \frac{Im\{\alpha_1\alpha_2\}}{\Omega}\right] \qquad (16)\\
&+ \frac{\epsilon_r}{N^2\Omega^2}\left[4Re\{\alpha_1\alpha_2\}\left(2+3\cos\left(\frac{\pi}{N}\right)+\cos\left(\frac{2\pi}{N}\right)\right)\right.\\
&+ \left. 2|\alpha_2|^2\left(1+2\cos\left(\frac{\pi}{N}\right)+\cos\left(\frac{2\pi}{N}\right)\right)+\frac{8|\alpha_3|^2}{9}\left(1+\cos\left(\frac{\pi}{N}\right)\right)\right]\\
&- \frac{\epsilon_i}{N^2\Omega^2}\left[4Im\{\alpha_1\alpha_2\}\left(1+2\cos\left(\frac{\pi}{N}\right)+\cos\left(\frac{2\pi}{N}\right)\right)\right] + O(|\epsilon_r,\epsilon_i|^2)
\end{aligned}
$$

Looking at these expressions when $\epsilon_r = \epsilon_i = 0$, we see that for ϵ_r, ϵ_i small enough, we are in the region $Re\{C\} < Re\{B\} < 0$ of Figure 1 (as

$$Re\{\beta_2\} - \frac{Im\{\alpha_1\alpha_2\}}{\Omega} = a < 0)$$

and thus there is a supercritical Hopf bifurcation to the $\tilde{\mathbf{Z}}_N$ oscillation as λ increases through $-2\epsilon_r[1+\cos(\pi/N)]$. The $\tilde{\mathbf{Z}}_N$ branch corresponds to two types of oscillation, depending on whether solutions of (11) and its symmetric counterpart are of the form $(w_p, w_q) = (w_p, 0)$ or $(w_p, w_q) = (0, w_q)$. Using (5) we see that if $w_p = 0$ then $z_j \sim \xi_-^{j-1} w_q$, i.e. $z_{j+1} = (\xi_-)z_j$, so the phase difference between neighbouring oscillators is $\left(1+\frac{1}{N}\right)\pi$. Similarly, if $w_q = 0$, then $z_j \sim \xi_+^{j-1} w_p$, i.e. $z_{j+1} = (\xi_+)z_j$, so the phase difference between neighbouring oscillators is $\left(1-\frac{1}{N}\right)\pi$. Thus we have shown that as λ increases through $-2\epsilon_r\,[1+\cos(\pi/N)]$ we have a supercritical bifurcation to two conjugate stable rotating waves, one rotating in each direction, both of which have the maximum possible magnitude of phase difference between neighbouring oscillators.

6 An aside

Looking at (15–16) we see that for ϵ_r, ϵ_i small we have $Re\{C\} \approx 2Re\{B\}$, and these are either both positive or both negative, depending on whether a is positive or negative, respectively. Looking at Figure 1, we see that the case $a > 0$ (corresponding to a subcritical Hopf bifurcation in an uncoupled oscillator) is uninteresting, in that the $\mathbf{D}_N$–symmetric Hopf bifurcation will not create any stable orbits. The only sector of interest in this respect is the one where

$$Re\{B\} < Re\{C\} < -Re\{B\}, \quad \text{for} \quad Re\{B\} < 0$$

because here we have the creation of either a stable $\mathbf{Z}_2(\kappa)$ orbit or a stable $\mathbf{Z}_2(\kappa, \pi)$ orbit in the double Hopf bifurcation. A simple rearrangement of the above expressions for $Re\{B\}$ (15) and $Re\{C\}$ (16) gives

$$\begin{aligned} Re\{C\} &= 2Re\{B\} \\ &+ \frac{\epsilon_r}{N^2\Omega^2}\left[2|\alpha_2|^2\left(\cos\left(\frac{2\pi}{N}\right)-1\right)+\frac{8|\alpha_3|^2}{9}\left(1-\cos\left(\frac{2\pi}{N}\right)\right)\right]+O(|\epsilon_r,\epsilon_i|^2), \end{aligned} \tag{17}$$

so by choosing various parameters correctly, it may be possible to push the normal form for the oscillators from the line $Re\{C\} = 2Re\{B\} < 0$ that we know we are on for $\epsilon_r = \epsilon_i = 0$ across the boundary $Re\{C\} = Re\{B\} < 0$ by increasing ϵ_r, as shown schematically in Figure 2 (compare with Figure 1, which shows the bifurcation diagrams in each sector). (Note that as N increases, the coefficient of the term in ϵ_r in (17) decreases, all other things being equal.) We demonstrate this transition below in an example for $N = 3$, the smallest number of oscillators to have the three different types of orbit created in a double Hopf bifurcation.

We are in the region $\epsilon_r > 0$, and we want to increase $Re\{C\}$, so we set $\alpha_2 = 0$. For simplicity we also set $\epsilon_i, \beta_1, \beta_3$ and β_4 to be zero. The example we use is

$$\dot{z}_j = (\lambda + 1.5i)z_j - 0.7z_j^2 + 2\overline{z}_j^2 - |z_j|^2 z_j + \epsilon(2z_j - z_{j+1} - z_{j-1}) \tag{18}$$

for $j = 1, 2, 3$ and the subscripts are taken mod 3, which corresponds to equation (1) with $\Omega = 1.5, \alpha_1 = -0.7, \alpha_3 = 2, \beta_2 = -1$ and $\epsilon_r = \epsilon$. For this system,

$$Re\{B\} = -\frac{1}{9} + \frac{\epsilon}{9\times 1.5^2}\left[\frac{4\times 2^2}{9}\left(\cos\left(\frac{\pi}{3}\right)+\cos\left(\frac{2\pi}{3}\right)\right)\right] + O(\epsilon^2) = -\frac{1}{9} + O(\epsilon^2)$$

so using (17), to first order in ϵ we should get a transition from Hopf bifurcation to a stable $\tilde{\mathbf{Z}}_3$ orbit to Hopf bifurcation to a stable $\mathbf{Z}_2$ orbit of some kind when

$$Re\{C\} - 2Re\{B\} = -Re\{B\}$$

i.e.

$$\frac{\epsilon}{N^2\Omega^2}\left[\frac{8|\alpha_3|^2}{9}\left(1-\cos\left(\frac{2\pi}{N}\right)\right)\right] = \frac{\epsilon}{9\times 1.5^2}\left[\frac{8\times 2^2}{9}\left(1-\cos\left(\frac{2\pi}{3}\right)\right)\right] = \frac{1}{9},$$

i.e. $\epsilon = 27/64 \approx 0.42$. That this transition does occur is demonstrated in Figure 3, where we show the type of orbit that is stable near the double Hopf bifurcation for (18) as a function of ϵ. This clearly shows that for $\epsilon <\sim 0.45$, there is a Hopf bifurcation to a stable $\widetilde{\mathbf{Z}}_3$ orbit, while for $\epsilon >\sim 0.45$, the bifurcation is to a stable $\mathbf{Z}_2(\kappa, \pi)$ orbit, in good agreement with the predicted behaviour.

7 Conclusion and further work

We have shown that if we take N identical oscillators, each of which undergoes a supercritical Hopf bifurcation as a parameter say, λ, increases, and couple them diffusively in a ring geometry with complex coupling constant $\epsilon_r + i\epsilon_i$, then (for small enough $|\epsilon_r|$ and $|\epsilon_i|$) the system as a whole undergoes a supercritical Hopf bifurcation to a stable rotating wave state as λ is increased, and that the magnitude of the phase difference between neighbouring oscillators is a maximum for $\epsilon_r > 0$ (i.e. π for N even and $(1 \pm 1/N)\pi$ for N odd) and a minimum (i.e. zero, corresponding to the synchronised state) for $\epsilon_r < 0$.

Further extensions could include coupling that is not restricted to nearest–neighbour, while still preserving the $\mathbf{D}_N$ symmetry of the system, or using nonlinear coupling, although we would need the coupling to have some linear component in order to split the eigenvalues of the Jacobian into single and double pairs.

Acknowledgements

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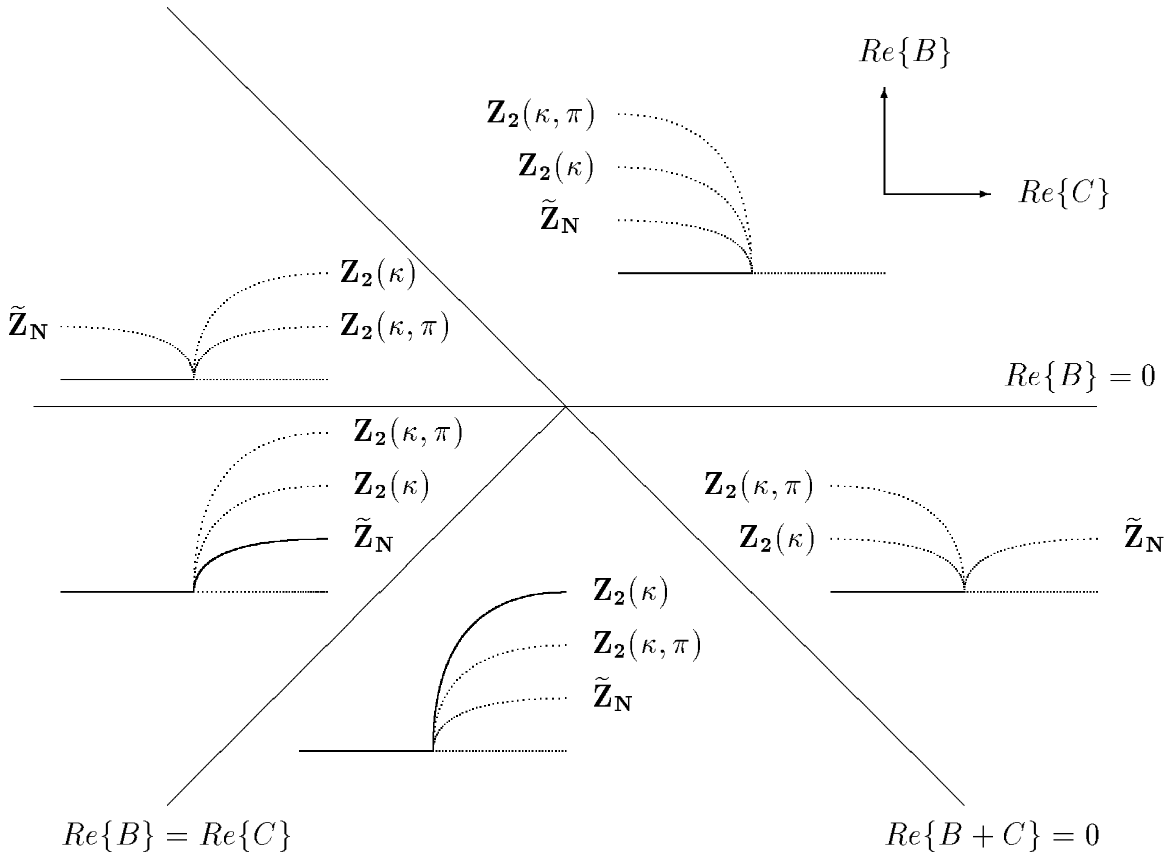

Figure 1: Generic bifurcation set for the normal form of the $\mathbf{D}_N$ symmetric Hopf bifurcation (10) for $N \geq 3$, $N \neq 4$, after Figure 3.1, Ch. XVIII of [6]. Within each sector of the $(Re\{C\}, Re\{B\})$ plane is a schematic bifurcation diagram with $Re\{\mu\}$ horizontally and some measure of the orbit vertically. Solid lines refer to stable solutions and dotted to unstable. Note that for any branch to be stable all must be supercritical, and then at most one branch is stable. Fifth–order terms in the normal form may interchange the $\mathbf{Z_2}(\kappa)$ and $\mathbf{Z_2}(\kappa, \pi)$ orbits. We have assumed that the origin is stable for $Re\{\mu\} < 0$. (We use "supercritical" and "subcritical" to refer to the direction in which a branch of orbits is created as $Re\{\mu\}$ is increased: supercritical branches are created as $Re\{\mu\}$ is increased, while subcritical are created as $Re\{\mu\}$ is decreased.)

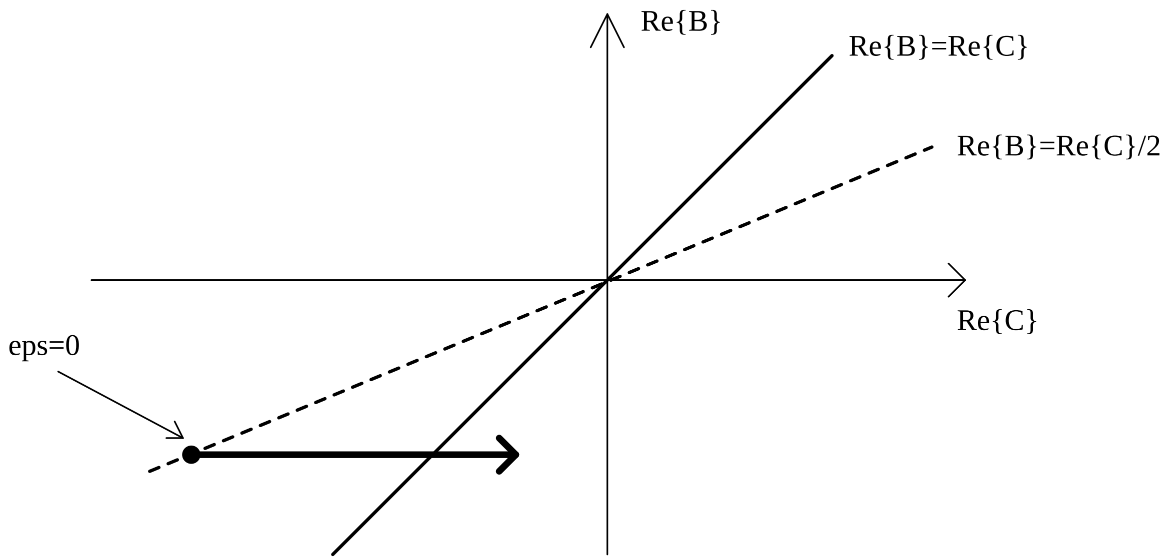

Figure 2: Schematic diagram showing how it might be possible to move from the line $Re\{C\} = 2Re\{B\} < 0$, which we know we are on at $\epsilon_r = \epsilon_i = 0$, across the line $Re\{B\} = Re\{C\} < 0$ by increasing ϵ_r. Compare with Figure 1, which shows bifurcation diagrams for the relevant sectors.

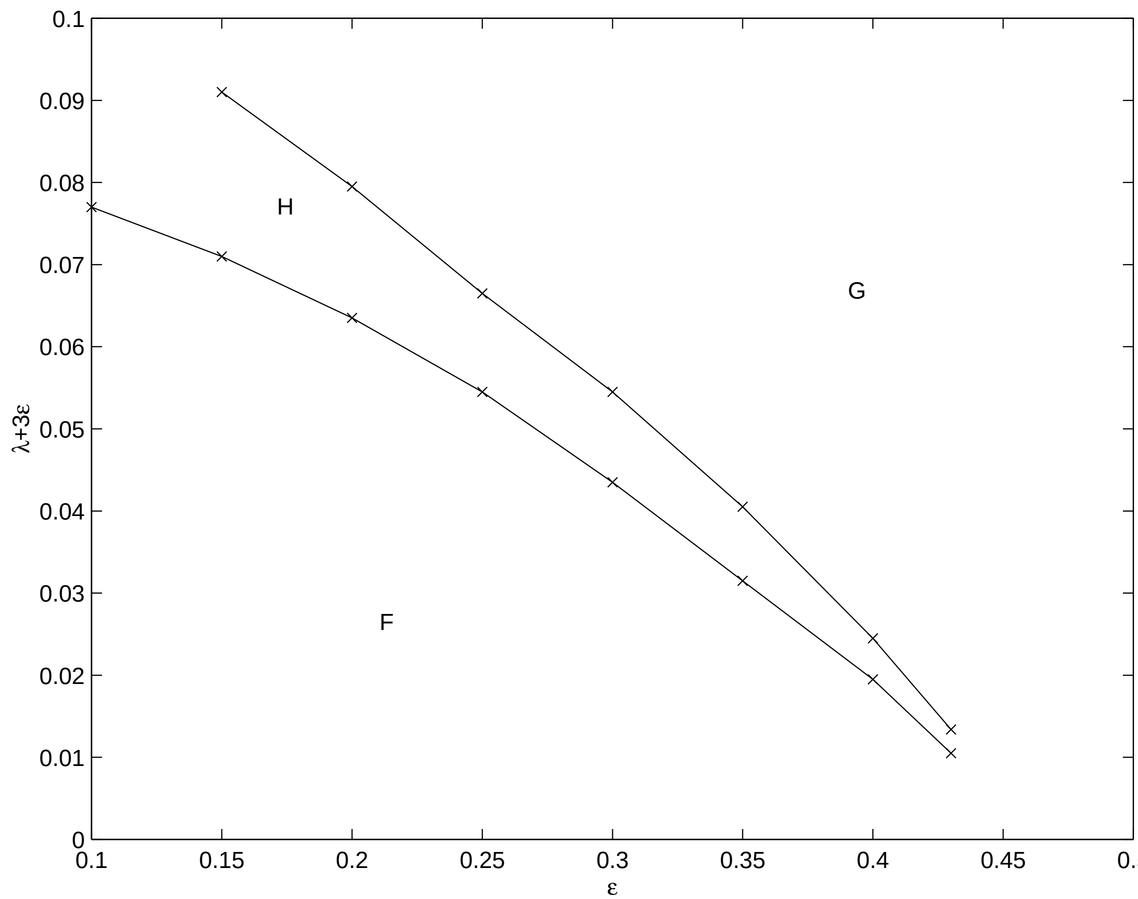

Figure 3: Transition from Hopf bifurcation to a stable $\tilde{\mathbf{Z}}_3$ orbit to Hopf bifurcation to a stable $\mathbf{Z}_2(\kappa, \pi)$ orbit as ϵ is varied in equation (18). In region F the $\tilde{\mathbf{Z}}_3$ orbit is stable, and in region G, the $\mathbf{Z}_2(\kappa, \pi)$ orbit is stable. There is non–periodic behaviour in the wedge H. The vertical coordinate is the distance in λ from the Hopf bifurcation.