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We have also found that this solution, whether regarded as a solution of the periodic problem or the Neumann problem, undergoes a supercritical blowout bifurcation. For the Neumann problem, this involves breaking only a reflectional symmetry of the solution while for the periodic problem, this is equivalent to a period-increasing bifurcation. We believe this to be the first observation of a blowout bifurcation from a chaotic solution of a PDE associated with a side–band instability.

For solutions with spatial period $2\pi/3$ and varying amounts of symmetry we found that chaotic solutions are always unstable with respect to perturbations of period 2π.

Acknowledgements

This work was supported by the EPSRC Applied Nonlinear Mathematics Initiative.

with respect to perturbations with period three times that of the solution, we would not expect to see this solution occurring starting with an arbitrary initial condition.

Finally, we consider solutions in $\mathrm{Fix}(\Sigma_4)$ and compute the dominant Lyapunov exponents associated with perturbations corresponding to the two two–dimensional irreducible representations of Σ_4 as a function of μ for $R = 80$, $\nu = 2$. We see for these parameter values that as μ is varied the solution in $\mathrm{Fix}(\Sigma_4)$ changes from periodic or quasiperiodic to chaotic but remains unstable with respect to both types of perturbation. An example of a chaotic solution from the parameter range shown in Fig. 13 is presented in Fig. 14

5 Conclusion and Discussion

In this paper we have extended the ideas in [5], regarding chaotic solutions with reflectional symmetries of the complex Ginzburg–Landau equation and their stability with respect to reflectional symmetry–breaking perturbations, to the study of the stability of such solutions with respect to perturbations having longer spatial wavelengths than the underlying solution.

Many solutions, as expected, are unstable with respect to perturbations of longer spatial wavelength. This indicates that the degree of self-organisation of chaotic solutions is very small compared to that for steady state and time periodic solutions. Indeed, the prospect of finding a spatio-temporal chaotic solution which is spatially periodic and stable with respect to *all* possible period-increasing perturbations is very unlikely. Thus, chaotic solutions which are spatially periodic are only found numerically because these conditions are imposed on them. We would not expect to see such solutions forming starting with an arbitrary, non-periodic initial condition. Thus, we conclude that while periodic boundary conditions are often mathematically very convenient, they are not necessarily physically relevant for chaotic solutions.

Homogeneous Neumann bounday conditions are often physically relevant and while the application of these boundary conditions would appear to restrict the symmetry of the problem it is well known that this problem can be embedded in the periodic problem which has much more symmetry [2, 8]. In this case, if we apply Neumann boundary conditions at $x = 0$ and $x = \pi$, then the solutions that we found in $\mathrm{Fix}(\Sigma_1)$ also have $A_x(\pi/2, t) = 0$ and are invariant under a reflection about $x = \pi/2$ and so satisfy

$$A(x, t) = A(\pi - x, t)$$

for all t. Moreover, these solutions are stable to perturbations which break this reflectional symmetry and so there is a degree of self-organisation in the solutions as they have a stable symmetric solution.

A chaotic solution in $\mathrm{Fix}(\Sigma_1)$ with spatial period π and homogeneous Neumann boundary conditions that is unstable with respect to perturbations of spatial period 2π, i.e. after the blowout bifurcation is shown in Fig. 6. A small perturbation with spatial period 2π is added at $t = 10$ and the symmetry of the solution is quickly lost as expected. The parameter values are $R = 4.2$, $\mu = -4$ and $\nu = 2.8$. Note that only the real part of the solution is shown.

Finally, we consider solutions in $\mathrm{Fix}(\Sigma_2)$ which have spatial period π, are even about the origin and are odd about $\pi/4$. Recall that in this case we are only interested in the Lyapunov exponents of multiplicity two associated with the two-dimensional irreducible representation of Σ_2. In Fig. 7 these dominant Lyapunov exponents are shown. We see that over the parameter range shown, the underlying solution changes from periodic or quasiperiodic to chaotic and back again but is always unstable with respect to perturbations of period 2π. We give an example of such an unstable chaotic solution in Fig. 8. A small perturbation with spatial period 2π is added to the solution at $t = 0.3$. The parameter values are $R = 62$, $\mu = -4$, $\nu = 2$.

For all contour plots, black contour lines correspond to negative values and grey contour lines to positive values.

4.2 Period $2\pi/3$ solutions

In order to investigate the effect of perturbations three times the period of the solution, we computed solutions with period $2\pi/3$, initially with no other symmetries imposed. The dominant Lyapunov exponents associated with the two-dimensional irreducible representation of $\mathbf{Z}_3$ are shown in Fig. 9. This shows a transition to chaos before and after which the solution is unstable with respect to perturbations of period 2π. An example of an unstable chaotic solution corresponding to this parameter range is shown in Fig. 10. Note that at approximately $t = 1.25$, the solution almost has $\mathbf{D}_3$ symmetry but then all symmetry is soon quickly lost after this point.

We next consider solutions in $\mathrm{Fix}(\Sigma_3)$ which have spatial period $2\pi/3$ and are also even about the origin. Again, we only consider the dominant (multiple) Lyapunov exponents associated with the two-dimensional irreducible representation which are shown in Fig. 11 as a function of R for $\mu = -4$ and $\nu = 2$. We see that for this range of parameters the underlying solution is either periodic, quasiperiodic, or chaotic, but is always unstable with respect to perturbations of period 2π. We show an example of such an unstable chaotic solution in Fig. 12 for parameter values $R = 9$, $\mu = -4$ and $\nu = 2$. Again, the black contour lines indicate negative values and the grey contour lines indicate positive values. We note that by rescaling the spatial scale the solution at these parameter values is the same as the solution which was stable with respect to perturbations of period twice that of the solution. Since this solution is not stable occasional "bursts" away from it.

The blowout bifurcation of Fig. 3 seems to be supercritical, as we see bursting behaviour at parameter values close to the bifurcation which is very similar to the on-off intermittency seen in many other examples of blowout bifurcations in low dimensional systems. In Fig. 5 we choose the parameter values $R = 4.2$, $\mu = -4$, $\nu = 2.1667$ and plot the norm of the vector formed from the odd–numbered Fourier coefficients in the spectral representation of the solution as a function of time. The norm is zero if and only if the solution satisfies $A(x,t) = A(x+\pi,t)$. The initial condition was randomly chosen and had spatial period 2π. Thus, for long periods of time, the chaotic motion appears to be even with period π while there are occasional bursts where the period is 2π.

We should also note that the blowout bifurcation does not occur at a particular parameter value but over a range of values. This is typical for a system in which the parameter we vary is *non–normal* [1, 7]. (A non–normal parameter is one for which not only the dynamics normal to the invariant subspace change as we vary the parameter, but also the dynamics restricted to the invariant subspace.)

The discovery of this blowout bifurcation is significant in that we believe it to be the first such bifurcation associated with side–band instabilities from an underlying spatio–temporally chaotic solution. Covas *et al* [7] found a blowout bifurcation in a PDE describing the dynamics of a mean field dynamo model, but in that case the instability acted to break a reflectional symmetry. Fujisaka *et al* [10] examined the stability of the spatially uniform solution of three PDEs with respect to spatially inhomogeneous perturbations and found on–off intermittency associated with blowout bifurcations. The advantage of examining the spatially uniform state is that an expression for the dominant normal Lyapunov exponent can then sometimes be explicitly derived.

The other curious feature of Fig. 3 is that the dominant Lyapunov exponents associated with the isotypic components W_3 and W_4 are very similar. In theory these quanitities are completely independent and so this similarity is somewhat surprising. We have investigated the solution to see whether it has any extra symmetries which we were not expecting and found none. Thus, we are unable to explain why these Lyapunov exponents are so similar.

The final observation for this example is that the dominant Lyapunov exponent associated with perturbations in W_2 is always zero indicating that the solution is stable with respect to these perturbations also. These perturbations are odd with period π and there is always a zero Lyapunov exponent associated with these perturbations as explained in [5]. Thus, within the space of 2π periodic functions, this chaotic solution which is even and has period π is stable with respect to all possible symmetry breaking perturbations.

4 Numerical results

In this section we describe some numerical results relating to the theory presented above. The results are obtained using a pseudo–spectral method as described in [5].

4.1 Period π solutions

Solutions with period π and no reflectional symmetries were computed together with the dominant Lyapunov exponents associated with the two isotypic components as a function of R for $\nu = 1.9$ and $\mu = -4$. Since $W_1 = \mathrm{Fix}(\mathbf{Z}_2)$, a positive dominant Lyapunov exponent in this case indicates a chaotic solution. As there are always three zero Lyapunov exponents associated with the motion in $\mathrm{Fix}(\mathbf{Z}_2)$ [5] then the dominant Lyapunov exponent associated with non-chaotic motion is always zero. The stability of this solution to period-doubling perturbations is determined by the dominant Lyapunov exponent associated with W_2. Numerical results are shown in Fig. 1. We see that for these parameter values there are intervals in which the solution is chaotic and stable to perturbations of period 2π (which we discuss below), periodic or quasiperiodic and unstable with respect to perturbations of period 2π, and chaotic and unstable with respect to perturbations of period 2π. A typical example of a chaotic solution that is unstable to perturbations of period 2π is shown in Fig. 2. Note that only the real part of the solution is shown. This is a contour plot with black contour lines for negative values and grey contour lines for positive values.

It was found that the chaotic solutions that are stable with respect to perturbations of period 2π in the interval containing $R = 4$ in Fig. 1 not only have period π but are also even about some point in $[0, \pi/2)$, i.e. they are conjugate to a solution in $\mathrm{Fix}(\Sigma_1)$ via a spatial translation. Thus, we computed these solutions in $\mathrm{Fix}(\Sigma_1)$ together with the two dominant Lyapunov exponents associated with the Σ_1-isotypic components W_3 and W_4. These are shown in Fig. 3 as a function of ν for $R = 4.2$ and $\mu = -4$. We see that for ν between 1.9 and approximately 2.15 the solution in $\mathrm{Fix}(\Sigma_1)$ is chaotic and stable with respect to perturbations of period 2π. A typical example of such a solution is shown in Fig. 4, corresponding to $\nu = 1.9$.

We see from Fig. 3 that as ν increases the solution in $\mathrm{Fix}(\Sigma_1)$ becomes unstable to perturbations of period 2π while remaining chaotic. This is known as a *blowout bifurcation* [7, 13] and has been studied widely in systems of coupled chaotic oscillators. Blowout bifurcations can be classified as either *subcritical* or *supercritical* [1]. The main difference is that for a subcritical bifurcation there are *riddled basins* of attraction before the bifurcation (when the normal Lyapunov exponent is negative) while for a supercritical bifurcation there is *on–off intermittency* after the bifurcation, where the attractor spends long periods close to the submanifold that was stable before the bifurcation with

Defining $\rho = \beta s_1$, we find that Σ_4 is generated by ρ and s_1 which satisfy

$$s_1^2 = I, \qquad \rho^6 = I, \qquad s_1\rho = \rho^{-1}s_1$$

and so it is isomorphic to $\mathbf{D}_6$. We note that $\beta = \rho s_1$ and $r_\omega = \rho^2$. Now $\mathbf{D}_6$ has four one–dimensional irreducible representations, corresponding to the four combinations of ρ and s_1 being $\pm I$, and two two–dimensional irreducible representations given by

$$\rho = \begin{bmatrix} \cos\omega/2 & -\sin\omega/2 \\ \sin\omega/2 & \cos\omega/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix} \quad \text{and } s_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{3.2}$$

and

$$\rho = \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \quad \text{and } s_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{3.3}$$

For all of the one–dimensional irreducible representations,

$$r_\omega = \rho^2 = I,$$

so these isotypic components are not of use for studying period–tripling instabilities. However, r_ω does not act trivially for either of the two–dimensional representations and so these are relevant. We note that for the underlying solution to be stable with respect to period-increasing perturbations, the dominant Lyapunov exponents associated with *both* the two-dimensional irreducible representations must be negative.

3.3 Higher values of n

By considering the cases of $n = 2$ and $n = 3$, the pattern for higher values of n can clearly be seen. Depending on the reflectional symmetries of the solutions, there are three basic cases to consider for each n:

(i) If the solution has no reflectional symmetries, then the solution is fixed only by $r_{2\pi/n}$ and so the group is $\mathbf{Z}_n$.

(ii) If the solution is either even or odd, then there is an additional reflectional symmetry so the group is $\mathbf{D}_n$.

(iii) If the solution is even about the origin and odd about $\pi/(2n)$, then the group is $\mathbf{D}_{2n}$.

Since all the dihedral and cyclic groups only have one and two dimensional irreducible representations, the methods used here are easily extended to higher values of n.

where $W_1 = \mathrm{Fix}(\mathbf{Z}_3)$. Since the two-dimensional irreducible representation is not absolutely irreducible, there is no further decomposition of the linear operator $g_A(A)$ into two diagonal blocks, as occurred in the previous section with the group $\mathbf{D}_4$. However, it does have a complex structure [15] in that

$$g_A(A)|_{W_2} = \begin{bmatrix} C & -D \\ D & C \end{bmatrix},$$

for some matrices C and D. This implies that if there is a solution $\phi = [u, v]^T$ of the variational equation

$$\dot{\phi} = g_A(A)|_{W_2}\phi, \tag{3.1}$$

then there is also another distinct solution of (3.1) given by $\phi = [-v, u]^T$. Thus the Lyapunov exponents are again of multiplicity two in this case.

The solution with spatial period $2\pi/3$ will be stable to perturbations of period 2π if the (multiple) dominant Lyapunov exponents associated with the isotypic component W_2 are negative.

We now consider solutions which have some reflectional symmetries and have period $2\pi/3$. If solutions are also even about the origin, then the solutions have symmetry group which we call Σ_3 generated by r_ω and s_1 and so is isomorphic to the dihedral group $\mathbf{D}_3$. This group has two one–dimensional irreducible representations $r_\omega = I, s_1 = I$ and $r_\omega = I, s_1 = -I$, and one two–dimensional representation

$$r_\omega = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \quad \text{and } s_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In a similar way to the $\mathbf{D}_4$ case above, r_ω acts as the identity for the one–dimensional irreducible representations and so perturbations in the corresponding isotypic components have the same period as the solution. Thus, only the two-dimensional irreducible representation is of interest and since it is also absolutely irreducible, the Lyapunov exponents associated with the corresponding isotypic component will have multiplicity two. Again numerically it is sufficient to consider only perturbations for which $s_1 = I$.

The theory is again similar for solutions which have spatial period $2\pi/3$ and are odd about the origin.

Finally, we consider solutions which are even about the origin, odd about $\pi/6$ and have period $2\pi/3$. It is helpful to define

$$\beta A(x,t) := s_2 s_1 r_{\pi/3} A(x,t) = -A(\pi/3 - x, t),$$

since functions fixed by β are odd about $\pi/6$. The symmetry group Σ_4 of these solutions thus includes s_1 (even about the origin), β (odd about $\pi/6$) and r_ω (period $2\pi/3$).

represent different combinations of reflectional symmetries being broken which preserve the period, which we considered in [5]. There is also one two–dimensional irreducible representation of $\mathbf{D}_4$ given by

$$R = \begin{bmatrix} \cos\pi/2 & -\sin\pi/2 \\ \sin\pi/2 & \cos\pi/2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad s_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

For this representation

$$r_\pi = R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

and so all perturbations in the corresponding isotypic component have minimal period 2π and so are spatially period doubling.

We note that since this two-dimensional irreducible representation is also absolutely irreducible, the linear operator $g_A(A)$ can be decomposed further into two identical blocks associated with the spaces on which s_1 acts as I or $-I$. This results in Lyapunov exponents of multiplicity two. Moreover, numerically it is sufficient to work with only one of these two identical blocks in order to find just one of the Lyapunov exponents. See Aston and Dellnitz [4] for more details. By choosing the block associated with the space on which $s_1 = I$, we have a Fourier decomposition of the perturbation in this case given by

$$A(x,t) = \sum_{k=1}^{\infty} b_k(t)\cos(2k-1)x + i\sum_{k=1}^{\infty} c_k(t)\cos(2k-1)x.$$

The alternative choice of perturbation which gives the second identical Lyapunov exponent consists of replacing the cosines with sines.

Stability of the chaotic solution with respect to period-doubling perturbations is determined by the sign of the dominant Lyapunov exponent associated with this type of perturbation.

3.2 Spatial period tripling ($n = 3$)

When $n = 3$, solutions have spatial period $2\pi/3$ and so are fixed by the action of $r_{2\pi/3}$. For ease of notation, we define $\omega = 2\pi/3$. If the solutions have no other symmetries, then they are contained in $\mathrm{Fix}(\mathbf{Z}_3)$. There are only two irreducible representations of $\mathbf{Z}_3$ which are given by

$$r_\omega = I, \qquad r_\omega = \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix}.$$

The corresponding isotypic decomposition is

$$X = W_1 \oplus W_2,$$

isotypic component. Thus,

$$A(x,t) \in W_1 \implies A(x,t) = \frac{b_0(t)}{2} + \sum_{k=1}^{\infty} b_k(t)\cos 2kx + i\left[\frac{c_0(t)}{2} + \sum_{k=1}^{\infty} c_k(t)\cos 2kx\right]$$

$$A(x,t) \in W_2 \implies A(x,t) = \sum_{k=1}^{\infty} b_k(t)\sin 2kx + i\sum_{k=1}^{\infty} c_k(t)\sin 2kx$$

$$A(x,t) \in W_3 \implies A(x,t) = \sum_{k=1}^{\infty} b_k(t)\cos{(2k-1)x} + i\sum_{k=1}^{\infty} c_k(t)\cos{(2k-1)x}$$

$$A(x,t) \in W_4 \implies A(x,t) = \sum_{k=1}^{\infty} b_k(t)\sin{(2k-1)x} + i\sum_{k=1}^{\infty} c_k(t)\sin{(2k-1)x}$$

We investigate solutions in $\mathrm{Fix}(\Sigma_1)$, and since we are interested in period–increasing instabilities we only calculate the dominant Lyapunov exponents associated with W_3 and W_4 (on which r_π acts as $-I$) and not that associated with W_2 (on which r_π acts as the identity).

It is also possible to consider solutions with period π which are odd functions of x. However, this is very similar to the previous case in that the symmetry of the solutions is again isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$ and so we do not consider this case in detail.

The other combination of reflectional symmetries, which we considered in [5], is when solutions are even about one point and odd about another. In particular, we consider solutions with a spatial period of π which are even about the origin and odd about $\pi/4$. To help in the following discussion, we define

$$\eta A(x,t) := s_2 s_1 r_{\pi/2} A(x,t) = -A(\pi/2 - x, t).$$

If $A(x,t)$ is fixed by η then it is odd about $\pi/4$. Thus, the group of symmetries of these solutions includes s_1 (even about the origin), η (odd about $\pi/4$) and r_π (period π).

Defining $R \equiv \eta s_1$ we see that this group, which we call Σ_2, is generated by R and s_1 which satisfy

$$R^4 = I, \quad s_1^2 = I, \quad s_1 R = R^{-1} s_1,$$

and so is isomorphic to $\mathbf{D}_4$. We note that $\eta = R s_1$ and $r_\pi = R^2$. There are four one–dimensional irreducible representations of $\mathbf{D}_4$ given by

$$\begin{array}{ll} R = I & s_1 = I \\ R = -I & s_1 = I \\ R = I & s_1 = -I \\ R = -I & s_1 = -I, \end{array}$$

but since $r_\pi == R^2 = I$ for all of these, perturbations in the corresponding isotypic components are not useful when considering spatial period doubling. Indeed, these

The Fourier decompositions of solutions in W_1 and W_2 are

$$\begin{aligned}
A(x,t) \in W_1 \Longrightarrow A(x,t) &= \frac{b_0(t)}{2} + \sum_{k=1}^{\infty}\{b_k(t)\cos 2kx + c_k(t)\sin 2kx\} \\
&\quad + i\left[\frac{d_0(t)}{2} + \sum_{k=1}^{\infty}\{d_k(t)\cos 2kx + e_k(t)\sin 2kx\}\right] \\
A(x,t) \in W_2 \Longrightarrow A(x,t) &= \sum_{k=1}^{\infty}\{b_k(t)\cos(2k-1)x + c_k(t)\sin(2k-1)x\} \\
&\quad + i\sum_{k=1}^{\infty}\{d_k(t)\cos(2k-1)x + e_k(t)\sin(2k-1)x\}
\end{aligned}$$

In this case, the solution with period π will be stable with respect to perturbations of period 2π if the dominant Lyapunov exponent associated with perturbations in W_2 is negative and unstable otherwise.

If we have solutions which have period π and in addition are even functions of x, then the solutions can be found by solving for A on the interval $[0, \pi/2]$ with homogeneous Neumann boundary conditions and are contained in $\mathrm{Fix}(\Sigma_1)$ where

$$\Sigma_1 = \{I, r_\pi, s_1, r_\pi s_1\},$$

which is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$ as r_π and s_1 are both generators for $\mathbf{Z}_2$. This group has four one–dimensional irreducible representations, corresponding to the four possible combinations of r_π and s_1 being $\pm I$, and there are four corresponding isotypic components given by

$$\begin{aligned}
W_1 &= \{A \in X : A_x(0,t) = 0 \text{ and } A_x(\pi/2,t) = 0\} = \mathrm{Fix}(\Sigma_1) \\
W_2 &= \{A \in X : A(0,t) = 0 \text{ and } A(\pi/2,t) = 0\} \\
W_3 &= \{A \in X : A_x(0,t) = 0 \text{ and } A(\pi/2,t) = 0\} \\
W_4 &= \{A \in X : A(0,t) = 0 \text{ and } A_x(\pi/2,t) = 0\},
\end{aligned}$$

We note that functions in each isotypic decomposition can be distinguished by different boundary conditions. This observation has been exploited numerically in bifurcation problems in [3].

By expanding $A(x,t)$ as a Fourier series it is easy to see which modes occur in each corresponding to, respectively, a rotation of the complex amplitude, space translation, time translation and a spatial reflection. We note that a special case of the rotation occurs when $\theta = \pi$ and this gives another symmetry of order two. As in [5], we define

$$\pi A(x,t) := s_2 A(x,t) = -A(x,t).$$

As we are interested in spatial period increasing bifurcations, we consider the CGL equation on the spatial domain $[0, 2\pi]$ together with periodic boundary conditions but we consider solutions with period $2\pi/n$ for some integer $n > 1$. Thus, perturbations with period 2π, the domain length, represent an increase in the period by a factor of n. Clearly such solutions are invariant under a translation of their period $2\pi/n$ and so are contained in $\mathrm{Fix}(\mathbf{Z}_n)$ where $\mathbf{Z}_n$ is the cyclic group of order n generated by $r_{2\pi/n}$. We will also consider solutions which have in addition some reflectional symmetries.

We noted in [5] that the CGL equation usually has three zero Lyapunov exponents. However, these are all associated with isotypic components which do not involve an increase in the period and so are not relevant in this context.

3 Period Increasing Bifurcations

We consider solutions with period $2\pi/n$ for particular values of n. We will concentrate on the values of $n = 2$ and $n = 3$ since then the generalisation to higher values of n will be obvious.

3.1 Spatial period doubling ($n = 2$)

When $n = 2$, the solutions that we are interested in have spatial period π and so are fixed by the action of r_π. If the solutions have no other symmetries then they are contained in $\mathrm{Fix}(\mathbf{Z}_2)$. The corresponding isotypic decomposition is simply

$$X = W_1 \oplus W_2$$

where

$$\begin{aligned} W_1 &= \{A \in X : r_\pi A = A\} = \mathrm{Fix}(\mathbf{Z}_2) \\ W_2 &= \{A \in X : r_\pi A = -A\}. \end{aligned}$$

that g satisfies the equivariance condition

$$\gamma g(A) = g(\gamma A) \quad \text{for all } \gamma \in \Gamma, \tag{2.2}$$

where Γ is a compact Lie group. For any subgroup Σ of Γ, we define the fixed point space

$$\mathrm{Fix}(\Sigma) = \{A \in X : \sigma A = A \text{ for all } \sigma \in \Sigma\}$$

and it is easily verified that

$$g : \mathrm{Fix}(\Sigma) \to \mathrm{Fix}(\Sigma)$$

for all subgroups Σ of Γ so that the fixed point spaces are invariant under the flow of the nonlinear equation (2.1).

For each subgroup Σ of Γ, there is a unique Σ-isotypic decomposition of the space X given by

$$X = \sum_k \oplus W_k,$$

where each isotypic component W_k is the sum of irreducible subspaces which are associated with one of the irreducible representations of Σ. If there is a solution $A(t) \in \mathrm{Fix}(\Sigma)$ of (2.1), then the Σ-isotypic components are invariant under the linearisation of g about $A(t)$, i.e.

$$g_A(A(t)) : W_k \to W_k$$

and so there is a block diagonal structure to the linear operator $g_A(A(t))$. Since this linear operator is involved in the variational equation which is used to compute Lyapunov exponents, we can associate Lyapunov exponents with a particular isotypic component. There are two important consequences of this decomposition which are as follows:

1. the Lyapunov exponents can be calculated for perturbations in each of the isotypic components independently;

2. the motion in $\mathrm{Fix}(\Sigma)$ will be stable if the dominant Lyapunov exponent associated with each of the isotypic components other than the trivial one (which is $\mathrm{Fix}(\Sigma)$) are negative.

We apply these ideas to the CGL equation (1.1) which has a number of symmetries given by

$$\begin{aligned}
\theta A(x,t) &= e^{i\theta} A(x,t), & \theta \in [0, 2\pi) \\
r_\alpha A(x,t) &= A(x+\alpha, t), & \alpha \in [0, 2\pi) \\
\tau_\beta A(x,t) &= A(x, t+\beta), & \beta \in \mathbb{R} \\
s_1 A(x,t) &= A(-x, t), &
\end{aligned}$$

odd perturbations.

In this paper we continue the investigation, but consider solutions that have a spatial period L and investigate their stability with respect to perturbations that have spatial period kL for some integer $k > 1$, i.e. *spatial period increasing* perturbations. These are often referred to as *side-band* perturbations. (Note that this is quite different from the ideas of period–doubling or multiplying that have gained much attention in the past 20 years — these refer to an increase in the *temporal* period of oscillation by a factor of 2 or more.) This is a generalisation of work by Benjamin and Feir [6] who considered the stability of travelling periodic water waves to side-band perturbations, and by Eckhaus [9] who considered the stability of spatially periodic steady state solutions of a PDE to side-band perturbations. This work was later extended and corrected by Stuart and DiPrima [14]. The work of Benjamin and Feir was in the context of a Hamiltonian system which was not the case for the work of Eckhaus. However, the concept of stability with respect to side-band perturbations is similar in both cases.

Fujisaka *et al* [10] also consider the stability of chaotic solutions of partial differential equations but they restrict attention to spatially uniform solutions and their stability with respect to non-uniform perturbations. This is analagous in some ways to bifurcation from a trivial (i.e. spatially uniform) solution. We take this process further by considering bifurcations from nontrivial (i.e. spatially non-uniform) solutions.

The CGL equation plays the role of a model partial differential equation to which we apply these ideas. However, this approach is of course very general and can be applied to a wide range of partial differential equations. Also, this approach can easily be generalised to higher spatial dimensions.

In Section 2, we describe the symmetries of the CGL equation and our approach to determining stability of these solutions by computing dominant Lyapunov exponents which are associated with particular isotypic components of the function space. We concentrate on spatial period doubling and tripling in Section 3 and show how these ideas generalise to larger period perturbations. Numerical results are presented in Section 4 while the significance of these results is discussed in Section 5.

2 Symmetry and Bifurcation

We briefly review our approach to dealing with symmetry breaking bifurcations in chaotic systems with symmetry for the sake of completeness. For more details, see [4, 5].

We consider a general evolution equation of the form

$$A_t = g(A), \qquad g : X \to X, \tag{2.1}$$

where g is assumed to be a nonlinear operator involving spatial derivatives and X is an appropriate Hilbert space which incorporates the boundary conditions. We also assume

1 Introduction

The formation of patterns in the solutions of partial differential equations which model many physical systems has been the subject of much interest over many decades. Associated with this are ideas of *self organisation* in which particular patterns are chosen by a particular system and this is determined by the stability of different patterns since only stable solutions will be seen in practice. Mathematically speaking, solutions of an equation are found in a particular function space. The question of stability can be a delicate one since it is often necessary to consider the effects of small perturbations on the solution which are not in the same space as the solution. A simple example is when a solution has certain symmetry properties but such a solution may be unstable to perturbations which break the symmetry of the solution.

Studies in pattern formation are usually concerned with either steady state or time periodic solutions of PDE's and patterns are often associated with symmetries of the solutions [11]. However, we consider patterns that occur in spatio-temporally chaotic solutions of PDE's, which are defined in terms of their symmetries, and of particular interest is their stability with respect to perturbations which break the symmetries of the solution. In a previous paper [5] we considered reflectional symmetries but in this paper, we consider symmetries which are associated with spatial periodicity.

Spatially periodic boundary conditions are often imposed on solutions of PDE's. This has the advantage of reducing an infinite spatial domain to a finite one. However, when considering the stability of such solutions, it is important to consider the effect of perturbations which are periodic, but which have a longer period than that of the solution itself. One example of this occurs in the Kuramoto-Sivashinsky equation in which there is a nontrivial branch of steady state solutions which bifurcates from the trivial solution. There are solutions on this branch which are stable with respect to periodic perturbations whose period is *any* integer multiple of the period of the solution (see the numerical results in [12]) and so we would expect to see this solution in the physical systems modelled by the Kuramoto-Sivashinsky equation.

In a previous paper [5] we investigated chaotic solutions of the complex Ginzburg–Landau (CGL) equation

$$A_t = RA + (1+i\nu)\nabla^2 A - (1+i\mu)A|A|^2, \qquad x \in [0, 2\pi) \tag{1.1}$$

with $A \in \mathbb{C}$ and $R, \nu, \mu \in \mathbb{R}$ that possessed various reflectional symmetries, concentrating on their stability with respect to perturbations without these symmetries. We found that for most parameter values, chaotic solutions that were restricted to lie within symmetric subspaces were unstable with respect to perturbations out of these subspaces. However, we did find a small region of parameter space in which there were solutions that were even about some point in the domain $[0, 2\pi)$ and were stable with respect to

Symmetry and Chaos in the Complex Ginzburg–Landau Equation. II: Translational symmetries

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March 11, 1999

Abstract

The complex Ginzburg–Landau (CGL) equation on a 1–dimensional domain with periodic boundary conditions has a number of different symmetries, and solutions of the CGL may or may not be fixed by the action of these symmetries. In this paper we investigate the stability of chaotic solutions that are spatially periodic but have a period that is some fraction of the domain length, L, with respect to perturbations that have a spatial wavelength equal to the domain length. We do this by considering the isotypic decomposition of the space of solutions and finding the dominant Lyapunov exponent associated with each isotypic component.

We find a region of parameter space in which there exist chaotic solutions with spatial period $L/2$ and homogeneous Neumann boundary conditions that are stable with respect to perturbations of period L. On varying the parameters in the CGL it is possible to arrange for this solution to become unstable to perturbations of period L while remaining chaotic, leading to a supercritical *blowout* bifurcation.

For a large number of parameter values checked, chaotic solution with spatial period $L/3$ were found to be unstable with respect to perturbations of period L.

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