160.734 Mid-Semester Test 2013

Time allowed: **55 minutes**.
Show **all** your working.

1. [**10 marks**]

   (a) Let $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 3 & 0 \end{bmatrix}$. Compute e^{tA}, for $t \in \mathbb{R}$.
   Hint: A is nilpotent.

   (b) Suppose that for $t \in \mathbb{R}$, $x(t)$, $y(t)$ and $z(t)$ are given by

   $$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -4 & 0 & 0 \\ 1 & -4 & 0 \\ 2 & 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad \begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}.$$

   Using your answer from part (a), derive explicit expressions for $x(t)$, $y(t)$ and $z(t)$ in terms of x_0, y_0 and z_0.

2. [**10 marks**] Consider the one-dimensional system, $\dot{x} = \sqrt{|x|}$, $x(0) = -1$.

   (a) Compute the solution to the system valid while $x(t) \leq 0$, and show that $x(2) = 0$.

   (b) For $t \geq 2$ the solution is not unique.
   Determine the range of values that are possible for $x(3)$.

3. [**10 marks**] Consider the system

   $$\begin{aligned} \dot{x} &= y + (x+3)(x-2) \\ \dot{y} &= xy \end{aligned}$$

   (a) Find all equilibria.

   (b) Classify each equilibrium as one of: *stable node*, *stable focus*, *unstable focus*, *unstable node*, *saddle*, or *other*.

   (c) Show that the system has a heteroclinic orbit.

4. [**10 marks**] Consider the system

   $$\begin{aligned} \dot{x} &= 3x + y + 2xy \\ \dot{y} &= 5xy - xy^2 \end{aligned}$$

   (a) Calculate $W^c(0,0)$ to third order.

   (b) Derive an ODE for the dynamics on $W^c(0,0)$ to third order.